1 February 2, 2018

Topics: verifying solution to a PDE, dispersion relations, well-posedness, general solution via integration.

1. (EPDE, Exercise 1.6) Verify the general solution of the heat equation on the real line:

\[ \begin{cases} 
    u_t = ku_{xx}, & -\infty < x < \infty, \ t > 0 \\
    u(x,0) = g(x) 
\end{cases} \]

is given by

\[ u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-s)^2/4kt} g(s) \, ds. \] (1)

2. (PDE p.51-52) Solve the heat equation with the initial condition \( u(x,0) = g(x) = e^{-x} \). To do so, use (1) and the integral identity \( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = 1 \) to show that the solution is

\[ u(x,t) = e^{kt-x}. \] (2)

3. Find the dispersion relation of the linear PDE:

\[ u_t = -u - \delta u_{xx} - u_{xxxx}, \ \delta > 0. \]

4. (Well-posed and ill-posed problems) Consider the following PDEs for \( u \) and \( v \):

\( \Box \) \begin{align*}
    u_{tt} - u_{xx} &= 0 \\
    u(x,0) &= 0 \\
    u_t(x,0) &= 0
\end{align*}

and

\( \Delta \) \begin{align*}
    v_{tt} - v_{xx} &= 0 \\
    v(x,0) &= 0 \\
    v_t(x,0) &= \epsilon \sin(\frac{\pi}{\epsilon})
\end{align*}

(3)

It is straightforward to check that \( u(x,t) = 0 \) and \( v(x,t) = \epsilon^2 \sin(\frac{\pi}{\epsilon}) \sin(\frac{t}{\epsilon}) \). For small \( \epsilon > 0 \), note that \( \Delta \) is a small perturbation to \( \Box \) in the initial derivative data. Use the notion of “stability with respect to initial data” to argue that \( \Delta \) is well-posed. (Hint: if \( \|v_t(x,0) - u_t(x,0)\| \leq \epsilon \), then \( \|v(x,t) - u(x,t)\| \leq \epsilon^2 \), for all \( t > 0 \).)
Similarly, consider

\[
\begin{aligned}
\Box \left\{ 
\begin{array}{l}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \\
\frac{\partial u}{\partial x} (x,0) = 0 \\
\frac{\partial u}{\partial t} (x,0) = 0
\end{array}
\right. \\
\text{and} \\
\triangledown \left\{ 
\begin{array}{l}
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = 0 \\
\frac{\partial v}{\partial x} (x,0) = 0 \\
\frac{\partial v}{\partial t} (x,0) = \epsilon \sin(\frac{x}{\epsilon})
\end{array}
\right.
\end{aligned}
\]  

(4)

Note that \(u(x,t) = 0\) and \(v(x,t) = \epsilon^2 \sin(\frac{x}{\epsilon}) \sinh(\frac{t}{\epsilon})\). Argue that (\(\triangledown\)) is an ill-posed problem.

5. Find the general solution of the PDE: \(u_{xt} + 3u_x = 1\).

2 February 9, 2018

Topics: conservations laws, differential operators in polar/spherical coordinate, BVP and eigenvalues/eigenfunctions, method of characteristics.

1. \((1D \text{ gas dynamics})\) Consider the one dimensional, time-dependent flow of gas under the assumption of constant pressure \(p\). If \(u\) denotes the gas velocity, \(\rho\) the density and \(e\) the energy per unit volume, the basic equations of gas dynamics are

\[
\begin{align*}
\frac{\partial u}{\partial t} + uu_x &= 0 \\
\frac{\partial \rho}{\partial t} + u\rho_x + \rho u_x &= 0 \\
\frac{\partial e}{\partial t} + ue_x + eu_x + pu_x &= 0.
\end{align*}
\]

(5, 6, 7)

Rewrite the above equations in terms of a conservation of law. In other words, find the quantity (vector) \(U\) and flux \(F(U)\) so that the above equations can be written in the form:

\[
U_t + [F(U)]_x = 0
\]

(8)

2. Suppose that \(u(x,y)\) satisfies the two-dimensional Laplace equation \(\nabla^2 u = 0\) and \(u\) only depends on the distance from the origin \(r = \sqrt{x^2 + y^2}\), i.e., \(u(x,y) \equiv v(r,t)\). Show that

\[
v_{rr} + \frac{1}{r}v_r = 0.
\]

(9)

3. For what value(s) of \(\lambda\) does the following linear system \((A - \lambda I)x = 0\) have a unique solution or infinitely many solutions, where

\[
A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}.
\]

Now consider the following two-point \(\text{Boundary Value Problem}\) (BVP) for \(0 < x < 2\pi\):

\[
\begin{align*}
u''(x) + \lambda u &= 0, \\
u(0) &= u'(2\pi) = 0.
\end{align*}
\]

(10, 11)

For what value(s) of \(\lambda\) does the BVP has a unique solution? Infinitely many solutions?
4. (APDE, §1.2, Exercise 8, p.26) Solve the initial value problem

\[ u_t + u_x - 3u = t, \quad x \in \mathbb{R}, \ t > 0, \tag{12} \]
\[ u(x, 0) = x^2, \quad x \in \mathbb{R}. \tag{13} \]

5. (APDE, §1.2, Example 1.10, p.18) Find the general solution to \( u_t + 2tu_x = 0. \)