MATH-UA 263 Partial Differential Equations
Recitation Summary

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Office Hour: Wednesday 2-4pm, WWH 1003

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1 February 2, 2018

Topics: verifying solution to a PDE, dispersion relations, well-posedness, general solution via integration.

1. (EPDE, Exercise 1.6) Verify the general solution of the heat equation on the real line:
$$\begin{cases} u_t = ku_{xx}, \quad -\infty < x < \infty,\, t > 0 \\ u(x,0) = g(x) \end{cases}$$
is given by
$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-s)^2/4kt} g(s)\, ds. \tag{1}$$

2. (PDE p.51-52) Solve the heat equation with the initial condition $u(x,0) = g(x) = e^{-x}$. To do so, use (1) and the integral identity $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = 1$ to show that the solution is
$$u(x,t) = e^{kt-x}. \tag{2}$$

3. Find the dispersion relation of the linear PDE: $u_t = -u - \delta u_{xx} - u_{xxxx}$, $\delta > 0$.

4. (Well-posed and ill-posed problems) Consider the following PDEs for u and v:
$$(\square) \begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases} \quad \text{and} \quad (\triangle) \begin{cases} v_{tt} - v_{xx} = 0 \\ v(x,0) = 0 \\ v_t(x,0) = \epsilon \sin(\frac{x}{\epsilon}) \end{cases} \tag{3}$$
It is straightforward to check that $u(x,t) = 0$ and $v(x,t) = \epsilon^2 \sin\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right)$. For small $\epsilon > 0$, note that $(\triangle)$ is a small perturbation to $(\square)$ in the initial derivative data. Use the notion of "stability with respect to initial data" to argue that $(\triangle)$ is well-posed. (*Hint*: if $\|v_t(x,0) - u_t(x,0)\| \le \epsilon$, then $\|v(x,t) - u(x,t)\| \le \epsilon^2$, for all $t > 0$.)

Similarly, consider

$$(\square)\begin{cases} u_{tt}+u_{xx}=0 \\ u(x,0)=0 \\ u_t(x,0)=0 \end{cases} \quad \text{and} \quad (\triangle)\begin{cases} v_{tt}+v_{xx}=0 \\ v(x,0)=0 \\ v_t(x,0)=\epsilon\sin(\frac{x}{\epsilon}) \end{cases} \tag{4}$$

Note that $u(x,t)=0$ and $v(x,t)=\epsilon^2\sin\left(\frac{x}{\epsilon}\right)\sinh\left(\frac{t}{\epsilon}\right)$. Argue that $(\triangle)$ is an ill-posed problem.

5. Find the general solution of the PDE: $u_{xt}+3u_x=1$.

2 February 9, 2018

Topics: conservations laws, differential operators in polar/spherical coordiates, BVP and eigenvalues/eigenfunctions, method of characteristics.

1. (*1D gas dynamics*) Consider the one dimensional, time-dependent flow of gas under the assumption of constant pressure p. If u deontes the gas velocity, ρ the density and e the energy per unit volume, the basic equations of gas dynamics are

$$u_t+uu_x=0 \tag{5}$$
$$\rho_t+u\rho_x+\rho u_x=0 \tag{6}$$
$$e_t+ue_x+eu_x+pu_x=0. \tag{7}$$

Rewrite the above equations in terms of a conservation of law. In other words, find the quantity (vector) $\boldsymbol{U}$ and flux $\boldsymbol{F}(\boldsymbol{U})$ so that the above equations can be written in the form:

$$\boldsymbol{U}_t+[\boldsymbol{F}(\boldsymbol{U})]_x=\boldsymbol{0}. \tag{8}$$

2. Suppose that $u(x,y)$ satifies the two-dimensional Laplace equation $\nabla^2 u=0$ and u only depends on the distance from the origin $r=\sqrt{x^2+y^2}$, ie, $u(x,y)\equiv v(r,t)$. Show that

$$v_{rr}+\frac{1}{r}v_r=0. \tag{9}$$

3. For what value(s) of λ does the following linear system $(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{x}=\boldsymbol{0}$ have a unique solution or infinitely many solutions, where

$$\boldsymbol{A}=\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}.$$

Now consider the following two-point *Boundary Value Problem* (BVP) for $0<x<2\pi$:

$$u''(x)+\lambda u=0, \tag{10}$$
$$u(0)=u'(2\pi)=0. \tag{11}$$

For what value(s) of λ does the BVP has a unique solution? Infinitely many solutions?

4. (APDE, §1.2, Exercise 8, p.26) Solve the initial value problem

$$u_t + u_x - 3u = t, \quad x \in \mathbb{R},\, t > 0, \tag{12}$$
$$u(x,0) = x^2, \quad x \in \mathbb{R}. \tag{13}$$

5. (APDE, §1.2, Example 1.10, p.18) Find the general solution to $u_t + 2tu_x = 0$.

3 February 16, 2018

Topics: first-order PDEs and method of characteristics.

1. Revisit APDE §1.2, Exercise 8, solve by two approaches: (1) using a suitable choice of coordinate transformation, and, (2) using the method of characteristics.

2. Solve the following PDE by method of characteristics. Sketch some characteristic curves.

$$\begin{cases} y^2 u_x + u_y = 0, \\ u(x,0) = x^2. \end{cases} \tag{14}$$

3. APDE, §1.2, Example 1.11.

4. Classify and solve the following PDE

$$\begin{cases} 2u_{xx} + 5u_{xt} + 3u_{tt} = 0, \\ u(x,0) = 0,\ u_t(x,0) = xe^{-x^2}. \end{cases} \tag{15}$$

4 February 23, 2018

Topics: Wave equation, d'Alembert's formula, domain of dependence, Heat equation and Cauchy problem.

1. Review the solution of #5 in HW3.

2. (*The hammer blow*, PDE 2.1 Exercise #5,6) Consider the wave equation $u_{tt} = u_{xx}$ on the entire real line $-\infty < x < \infty$ with zero initial position $u(x,0) = 0$ and initial velocity $u_t(x,0) = g(x)$, where $g(x) = 1$ for $|x| < 1$ and $g(x) = 0$ for $|x| \geq 1$. Sketch the solution at time instants $t = \frac{1}{2}, 1, \frac{3}{2}, 2$ and $t = \frac{5}{2}$. What is the maximum displacement $\max_x u(x,t)$?

3. Solve the heat equation on the whole line with given initial data $u(x,0) = \phi(x)$:

   (a) $\phi(x) = 1$ for $|x| < l$, and $\phi(x) = 0$ for $|x| > l$.

   (b) $\phi(x) = x^2$.

   (c) $\phi(x) = e^{3x}$.

5 March 2, 2018

Topics: Duhamel's princple for the wave, heat, and linear advection equations.

1. Review the solution of #4 in HW3.
2. Principle of superposition for breaking down the solution of the inhomogeneous equation (with sources) with nonzero initial data into two subproblems: one homogeneous problem with nonzero initial data, and one inhomogeneous problem with zero initial data (Duhamel's principle).
3. Explain Duhamel's principle for the wave, heat, and linear advection equations.
4. Exercise 1, §3.4, PDE.
5. Part of #6 in HW4.
6. APDE, §2.5, Exercise #3 and #4.

6 March 9, 2018

Topics: Review midterm soluctions.

7 March 16, 2018

Spring break.

8 March 23, 2018

Topics: Review #3, #4 on the midterm, linear PDEs in bounded domains, separation of variables.

1. Use separation of variables to find the solution to the heat equation with homogeneous Neumann BCs:

$$\begin{aligned} u_t &= k u_{xx}, \quad \text{for } \ 0 < x < L,\, t > 0, \\ u(x,0) &= \phi(x), \\ u_x(0,t) &= u_x(L,t) = 0. \end{aligned}$$

2. Use separation of variables to find the solution to the wave equation with homogeneous Dirichlet BCs:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad \text{for } \ 0 < x < L,\, t > 0, \\ u(x,0) &= f(x), \\ u_t(x,0) &= g(x), \\ u(0,t) &= u(L,t) = 0. \end{aligned}$$

9 March 30, 2018

Topics: separation of variables and Fourier series.

1. Consider
$$X''(x) + \lambda X(x) = 0, \quad 0 < x < L. \tag{16}$$
For each of the following BCs, find its eigenvalues and eigenfunctions $\{\lambda_n, \varphi_n\}$ to the above 2-point BVP.

BCs	λ_n	φ_n
Dirichlet: $X(0) = X(L) = 0$		
Neumann: $X'(0) = X'(L) = 0$		
Mixed: $X'(0) = X(L) = 0$		
Mixed: $X(0) = X'(L) = 0$		
Periodic: $X(0) = X(L), X'(0) = X'(L)$		

2. Solve the following heat equation with initial condition and boundary conditions:
$$\begin{aligned} u_t &= k u_{xx}, \quad \text{for } \ 0 < x < L,\, t > 0, \\ u(x,0) &= 1 + \frac{1}{2018}\cos\left(\frac{2018\pi x}{L}\right), \\ u_x(0,t) &= u_x(L,t) = 0. \end{aligned}$$

3. Solve the wave equation with Dirichlet BCs:
$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad \text{for } \ 0 < x < L,\, t > 0, \\ u(x,0) &= f(x), \ u_t(x,0) = g(x), \\ u(0,t) &= u(L,t) = 0. \end{aligned}$$
Use Fourier series to express the solution.

10 April 6, 2018

1. Use separation of variables to solve the wave equation with Neumann BCs and initial conditions:
$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad \text{for } \ 0 < x < 1,\, t > 0, \\ u(x,0) &= f(x), \ u_t(x,0) = 0, \\ u_x(0,t) &= u_x(1,t) = 0, \end{aligned}$$
where $f(x)$ is a "bump" function
$$f(x) = \begin{cases} \cos(4\pi x) & \frac{3}{8} < x < \frac{5}{8}, \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$
Plot the solution by using MATLAB, Python, Maple or Mathematica, etc.

2. Pointwise, uniform convergence of Fourier series. Convergence in L^2-norm, Parseval's identity, and and completeness of an orthogonal set (notion of orthonormal basis).

11 April 13, 2018

Topics: inhomogeneous BCs and sources in bounded domains, Sturm-Liouville theory.

1. (APDE Example 4.25) Consider the heat equation with source:

$$\begin{aligned} u_t - ku_{xx} &= f(x,t), \quad \text{for } \ 0 < x < \pi, \ t > 0, \\ u(0,t) &= u(\pi,t) = 0, \\ u(x,0) &= 0. \end{aligned}$$

   Use Duhamel's principle to verify the "separation-of-variable" solution given in class.

2. Solve the heat equation wih source and nonzero IC:

$$\begin{aligned} w_t &= 3w_{xx} + 2e^{-t}(1-x), \quad \text{for } \ 0 < x < 1, \ t > 0, \\ w(0,t) &= w(1,t) = 0, \\ w(x,0) &= x^2 + x - 2. \end{aligned}$$

3. Solve the heat equation with inhomogeneous BCs:

$$\begin{aligned} u_t - 3u_{xx} &= f(x,t), \quad \text{for } \ 0 < x < 1, \ t > 0, \\ u(0,t) &= 2e^{-t}, \ u(1,t) = 1, \\ u(x,0) &= x^2. \end{aligned}$$

   Hint: after a suitable change of variable, you can reduce this problem to #2.

4. Sturm-Liouville theory, adjoint operators.

12 April 20, 2018

Topics: separation of variables, Poisson/Laplace equations.

1. Consider $u_t = u_{xx}$ for $-l < x < l$ with periodic BCs and $u(x,0) = \phi(x)$. A student asked whether taking the limit $l \to \infty$ would converge to the solution for the heat equation on the real line. My answer is, formally yes. But it is difficult to justify rigorously. Note that, for the Fourier coefficient $c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}dx$, it is nontrivial to take $l \to \infty$.

   However, you can derive the solution for the heat equation on the real line using *Fourier Transform*. You need some background knowledge on Fourier analysis and complex analysis (contour integrals). That's why this is not covered in this class. But it is essentially the same separation-of-variable technique we apply to bounded-domain problems.

2. In #4 of Homework 5, the solution to the PDE

$$u_t = u_{xx}, \quad \text{for } \; 0 < x < \frac{1}{2},\, t > 0,$$
$$u(0,t) = u_x(\frac{1}{2}, t) = 0,$$
$$u(x,0) = \begin{cases} 1 & \frac{3}{8} < x < \frac{1}{2}, \\ 0 & 0 < x < \frac{3}{8}. \end{cases}$$

is given by

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \cos\left(\frac{3(2n-1)\pi}{8}\right) e^{-((2n-1)\pi)^2 t} \sin((2n-1)\pi x). \tag{18}$$

You argue by symmetry that this solution also satisfies the extended problem:

$$u_t = u_{xx}, \quad \text{for } \; 0 < x < 1,\, t > 0,$$
$$u(0,t) = u(1,t) = 0,$$
$$u(x,0) = \begin{cases} 1 & \frac{3}{8} < x < \frac{5}{8}, \\ 0 & \text{elsewhere.} \end{cases}$$

However, if you solve the extended problem directly, you should get

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \left(\cos\left(\frac{3n\pi}{8}\right) - \cos\left(\frac{5n\pi}{8}\right)\right) e^{-n^2\pi^2 t} \sin(n\pi x). \tag{19}$$

Question for thought: are (18) and (19) the same series?

3. Solve the Lapace equation in a retangle:

$$\Delta u = 0, \quad \text{for } \; 0 < x < a,\, 0 < y < b,$$
$$u_x(0,y) = u_x(a,y) = 0,$$
$$u(x,0) = f(x), \quad u_y(x,b) = g(x).$$

Hint: a choice of *good* basis function in the y-variable BVP can significantly simplify your work.

4. Poisson equation on a rectangle:

$$\Delta u = 1, \quad 0 < x < a,\, 0 < y < b,$$
$$u_x(0,y) = u_x(a,y) = 0,$$
$$u(x,0) = u(x,b) = 0.$$

13 April 27, 2018

Cancelled.

14 May 4, 2018

Office hour on May 9 (Wed) is moved to **2-4pm May 14 (Mon)**.

Review topics:

1. First-order PDEs, method of characteristics, change of coordinates.
2. Classification of second-order linear equations, especially, hyperbolic equations (coordinate transformation).
3. PDE on the real line (infinite domain):
   (a) Heat equation: general solution (Green's function). For some initial conditions, one can simplify the integral using completing-the-square technique. If the IC is also a Gaussian, one can easily simplify the integral by shifting the Gaussian (see HW6 #8).
   (b) Wave equation: d'Alembert formula.
   (c) Duhamel's principle!
4. PDE in bounded domain. A typical problem can be a combination of:
   - Equations: heat/wave/Laplace/Poisson + some variations.
   - BCs: homogeneous Dirichlet/Neumann/mixed/periodic/Sturm-Liouvilles, and nonhomogeneous BCs.
   - With IC or zero IC.
   - With/without source terms.
   - 1D or 2D.

   No matter how complicated a problem may look, the solution techniques remain the same: *Divide-and-Conquer* and *Separation-of-Variables*. Generally, divide a complex problem into one of the following 3 mini-problems:

   (a) (Steady-state) PDE with nonhomogeneous BCs, zero IC, no source term.
   (b) PDE with homogeneous BCs, with ICs, no source term.
   (c) PDE with source term, homogeneous BCs, and zero IC.

Some examples:

1. Solve $xu_t - tu_x = xt$, where $u(x,0) = 1 + x$ for $x \geq 0$.
2. Find the general solution to the PDE: $u_{tt} + u_{tx} - 2u_{xx} = t$.
3. Solve $u_t = ku_{xx}$ on $\mathbb{R}$ with $u(x,0) = e^{-x^2}$. Simplify the integral using the shifting argument.
4. Use separation of variables, find the solution to $-(u_{xx}+u_{yy})+2u_x = 0$ in the square domain $0 < x, y < 1$ with homogeneous Dirichlet BCs: $u(x,0) = u(x,1) = 0$,and $u(0,y) = (1-y)y$, and $u_x(1,y) = 0$.