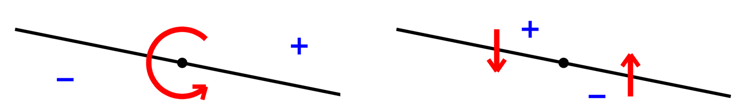
Unravelling the Resonant Instabilities of a Stratified Gravity Wave

Yuanxun Bill Bao & David J. Muraki
Department of Mathematics, Simon Fraser University, Burnaby, BC, Canada

Waves in a 2D Stratified Fluid

Boussinesq Fluid with Constant Stratification

$$\nabla \cdot \vec{\mathbf{u}} = 0 \quad ; \quad \frac{D\eta}{Dt} = -b_x \quad ; \quad \frac{Db}{Dt} = -\mathcal{N}^2 w$$

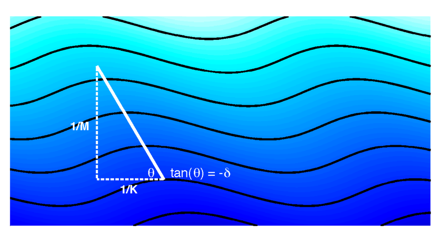


- Streamfunction, $\psi(x,z,t)$ & velocity, $\vec{\mathbf{u}} = (u,w) = (\psi_z, -\psi_x)$
- Buoyancy, $b(x,z,t)$ & vorticity, $\eta(x,z,t) = \psi_{zz} + \psi_{xx}$
- Brunt-Väisälä frequency, $\mathcal{N}$ & stable stratification

Exact Nonlinear Solution — Finite-Amplitude Gravity Wave

$$\begin{pmatrix} \psi \\ b \end{pmatrix} = \begin{pmatrix} -\Omega/KM \\ \mathcal{N}^2/M \end{pmatrix} 2\epsilon \, \sin(Kx + Mz - \Omega t)$$

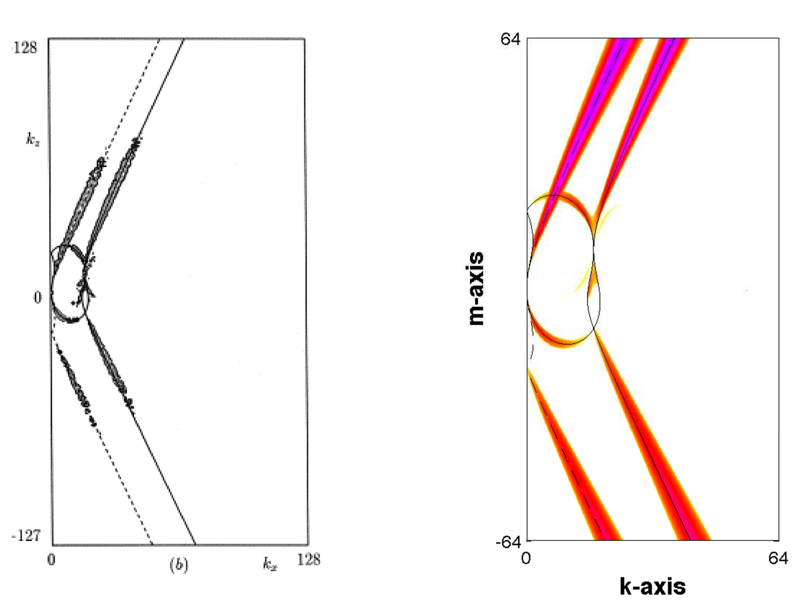
- Primary wavenumbers: (K, M)
- Propagation angle: $\delta = -\dfrac{K}{M}$



- Dispersion relation: $\Omega^2(K,M) = \dfrac{\mathcal{N}^2 K^2}{K^2 + M^2}$

Unstable Spectrum: DNS & Floquet

- Motivation: To compute unstable disturbance wavemodes using Floquet/Fourier approach
- Excited Fourier amplitudes of gravity wave disturbances evolved from a weak noise via Direct Numerical Simulations (Lin, 2000) — (left)
- Unstable wavemodes via Floquet/Fourier computation — (right)



References

[1] R.P. Mied, JFM, 78 (1976).
[2] K. Hasselmann, JFM, 30 (1977).
[3] P.G. Drazin, Proc. Roy. Soc. A, 356 (1977).
[4] J. Klostermeyer, Geophys. Astrophys. Fluid Dyn., 61 (1991).
[5] L.J. Sonmor & G.P. Klaassen, JFM, 324 (1996).
[6] P.N. Lombard & J.J. Riley, Dyn. Atmos. Oceans, 23 (1996).
[7] C.-L. Lin, Dyn. Atmos. Oceans, 32 (2000).

Linear Stability Analysis

- Linear stability analysis on dimensionless gravity wave + disturbances

$$\begin{pmatrix} \psi \\ b \end{pmatrix} = \begin{pmatrix} -\Omega \\ 1 \end{pmatrix} 2\epsilon \sin(x + z - \Omega t) + \begin{pmatrix} \tilde{\psi}(x,z,t) \\ \tilde{b}(x,z,t) \end{pmatrix}$$

- Dimensionless frequency: $\Omega^2 = \dfrac{1}{1+\delta^2}$
- Linear PDE system with non-constant, but periodic coefficients

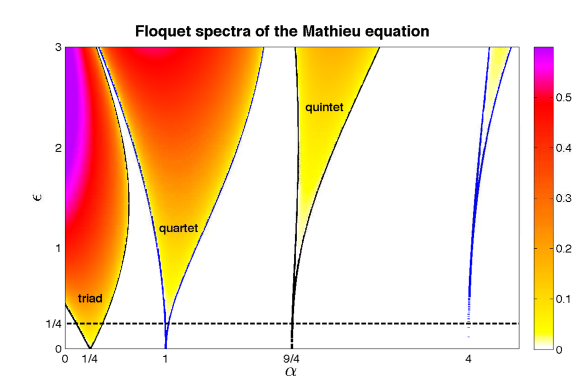
$$\begin{aligned} \tilde{\eta}_t + \tilde{b}_x - 2\epsilon J\big(\Omega\tilde{\eta} + \tilde{\psi}/\Omega \,,\, \sin(x+z-\Omega t)\big) &= 0 \\ \tilde{b}_t - \tilde{\psi}_x - 2\epsilon J\big(\Omega\tilde{b} + \tilde{\psi} \,,\, \sin(x+z-\Omega t)\big) &= 0 \\ \tilde{\psi}_{zz} + \delta^2 \tilde{\psi}_{xx} &= \tilde{\eta} \end{aligned}$$

- Linear advection in Jacobian:

$$J(f,g) = \begin{vmatrix} f_x & g_x \\ f_z & g_z \end{vmatrix} = f_x g_z - g_x f_z$$

Floquet Theory

Instabilities of the Mathieu Equation: $\ddot{u} + (\alpha + \epsilon \sin t)\, u = 0$



- Floquet representation with Fourier series:

$$u(t) = e^{-i\omega t} \left\{ \sum_{-\infty}^{+\infty} v_n \, e^{-int} \right\} = \begin{matrix}\text{exponential} \\ \text{part}\end{matrix} \times \left\{ \begin{matrix}\text{co-periodic} \\ \text{part}\end{matrix} \right\}$$

- Floquet exponent, $\omega(\alpha;\epsilon)$ & Im $\omega > 0 \to$ instability

Floquet/Fourier Analysis for PDEs

- Floquet representation with disturbance wavenumbers, (k, m)

$$\begin{pmatrix} \tilde{\psi} \\ \tilde{b} \end{pmatrix} = e^{i(kx + mz - \omega t)} \left\{ \sum_{-\infty}^{+\infty} \vec{v}_n \, e^{in(x+z-\Omega t)} \right\}$$

- Floquet exponent, $\omega(k,m;\epsilon)$ & Im $\omega > 0 \to$ instability
- Dispersion relation for disturbances: $\omega^{\pm}(k,m;0) = \pm \dfrac{|k|}{\sqrt{\delta^2 k^2 + m^2}}$
- A generalized eigenvalue problem with Hill's infinite matrix

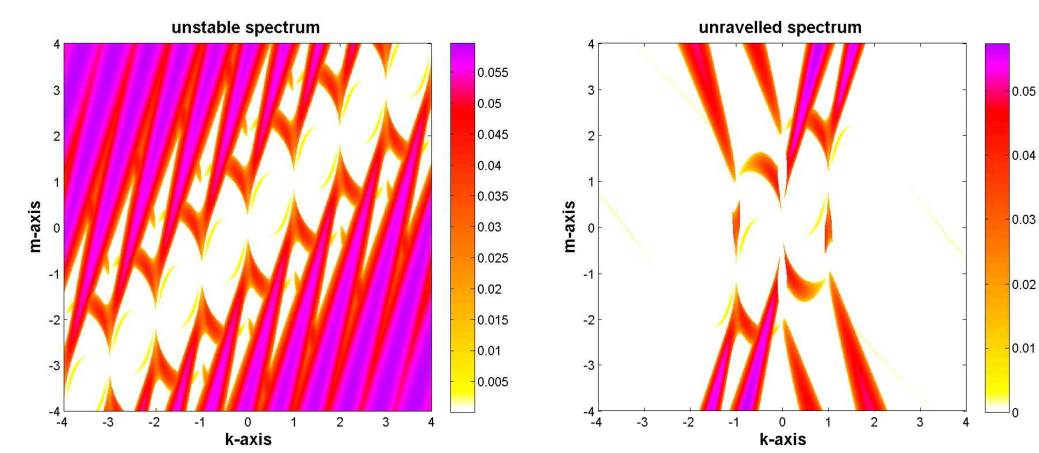
$$\begin{bmatrix} \ddots & \ddots & & \\ \ddots & \mathbf{S}_0 & \epsilon\mathbf{M}_1 & \\ & \epsilon\mathbf{M}_0 & \mathbf{S}_1 & \ddots \\ & & \ddots & \ddots \end{bmatrix} - \omega \begin{bmatrix} \ddots & & & \\ & \mathbf{\Lambda}_0 & & \\ & & \mathbf{\Lambda}_1 & \\ & & & \ddots \end{bmatrix}$$

- $2{\times}2$ $\mathbb{R}$ blocks: $\mathbf{M}_n(k,m)$; $\mathbf{S}_n(k,m)$, symmetric; $\mathbf{\Lambda}_n(k,m)$, diagonal
- Truncate $-N \le n \le N$ & compute $4N{+}2$ eigenvalues $\{\omega(k,m;\epsilon)\}$

A Tangle of Unstable Eigenvalues

- Unstable Floquet eigenvalues selected by maximum growth rate — (left)
- Periodicity from index shifts $\to$ multiple counting of Im ω's

$$\begin{pmatrix} \tilde{\psi} \\ \tilde{b} \end{pmatrix} = e^{i((k+q)x + (m+q)z - (\omega + \Omega q)t)} \left\{ \sum_{-\infty}^{+\infty} \vec{v}_{n+q} \, e^{in(x+z-\Omega t)} \right\}$$



- QUESTION 1: is there an association of $\omega(k,m;\epsilon)$ with instabilities corresponding to physical wave resonances — as in Lin (2000)?
- Yes! An unravelled spectrum of unstable Floquet eigenvalues — (right)
- QUESTION 2: are there two values of $\omega(k,m;\epsilon)$ as in the dispersion relation, $\omega^{\pm}(k,m;0)$; or $4N+2$ as from the truncated Hill's matrix?
- QUESTION 3: is the index-periodicity of the Floquet/Fourier method a numerical artifact, or a natural feature of Floquet theory?

Perturbative Analysis

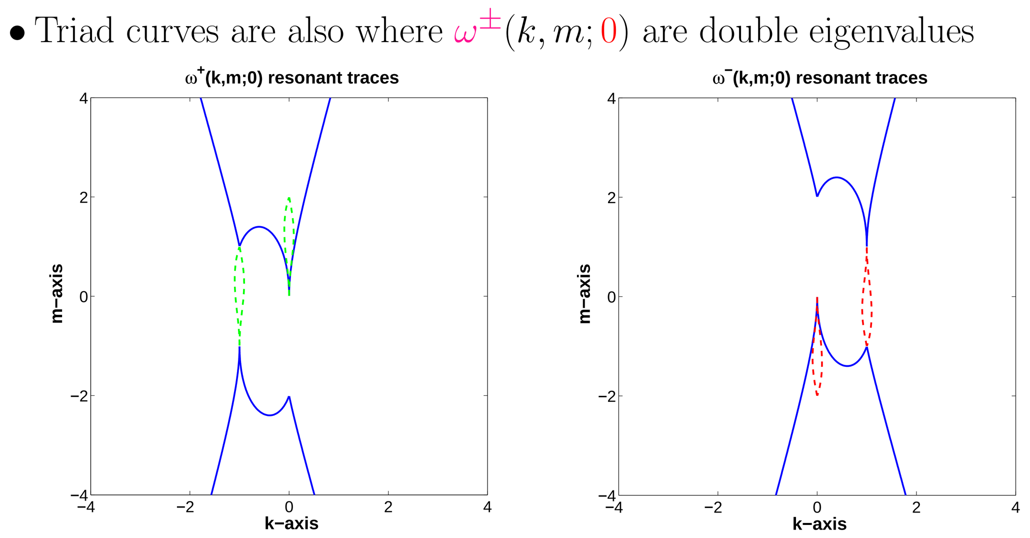
- Two branches from ϵ-perturbation theory: $\omega^{\pm}(k,m;\epsilon) \sim \omega^{\pm}(k,m;0)$
- Complex ω's arise from ϵ-perturbation of multiple Hill's eigenvalues

Triad ($n = \pm 1$) Resonance Analysis

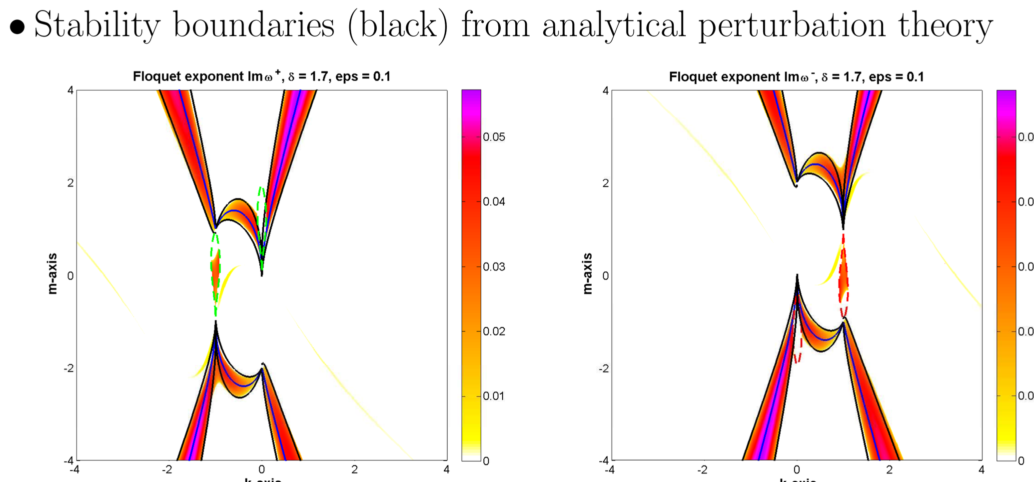
- (k,m) – curve for Triad resonances: ($+$, left & $-$, right)

$$\omega^{\pm}(k,m;0) = \omega^{\pm}(k+n, m+n; 0) - n\Omega$$

- Triad curves are also where $\omega^{\pm}(k,m;0)$ are double eigenvalues

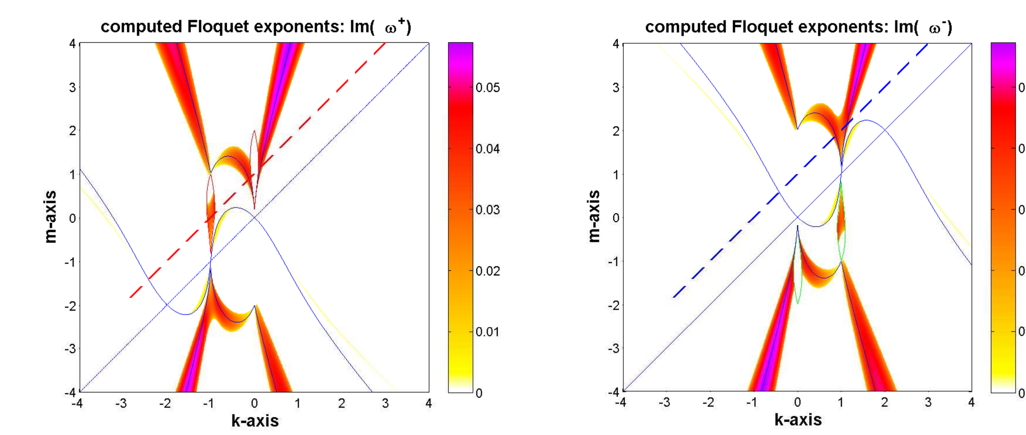


- Triad blue traces are unstable by perturbation theory
- Unravelled Im $\omega^{\pm}$ by ϵ–continuity: *computational perturbation theory*
- Stability boundaries (black) from analytical perturbation theory

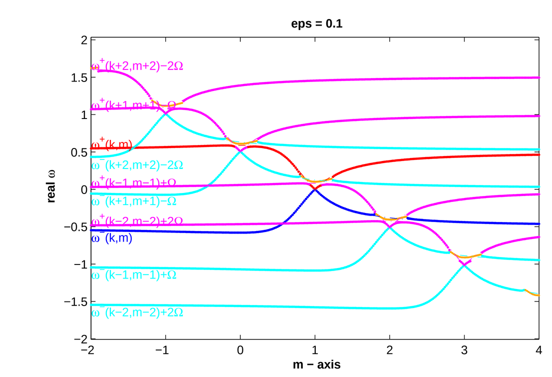


A Riemann Sheet Perspective ?

- Plot all Re $\omega^{\pm}$ along the cross-sectional cuts



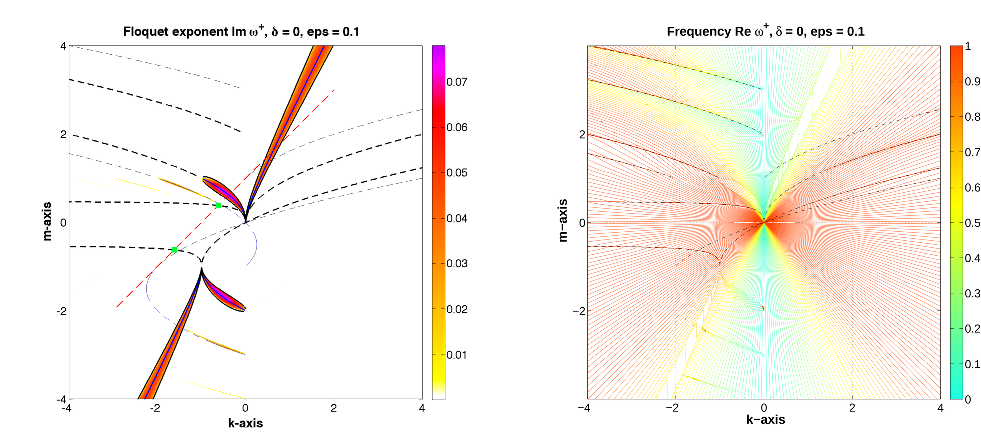
- Unravelled values of Re $\omega^{+}(k,m)$ & Re $\omega^{-}(k,m)$
- The other curves correspond to $\omega^{\pm}(k+n, m+n; \epsilon) - n\Omega$



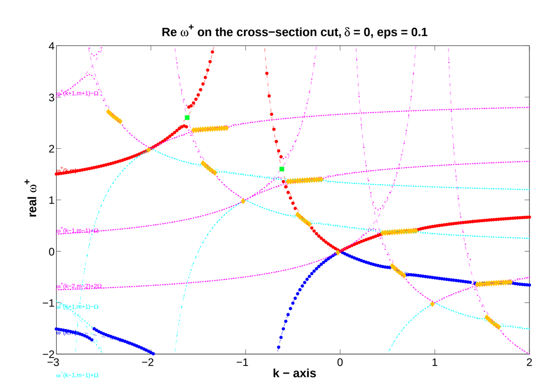
- Layered curves suggest a Riemann sheet interpretation

Hydrostatic Limit ($\delta = 0$)

- Untangled Im ω^{+} with triad (thick) & quartet (thin) traces — (left)
- Frequency Re ω^{+} suggests branch cuts along the stable traces — (right)



- Structure of Riemann sheets along the cross-sectional cut with matching instabilities & branch cuts



Preliminary Conjectures

- Branches of $\omega^{\pm}(k,m;\epsilon)$ need discontinuities associated with traces
- Floquet spectral theory suggests branch cuts for complex k & m
- Complications at high-multiplicity eigenvalues $\to$ crossing traces