An Immersed Boundary Method with Divergence-Free Velocity Interpolation

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A 2D pumping membrane

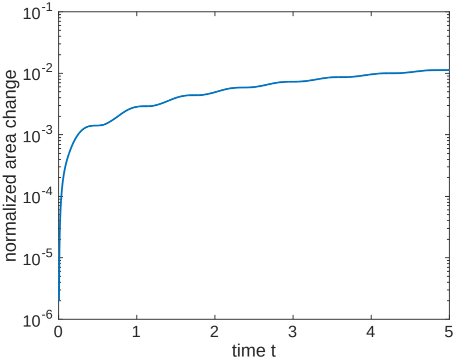
- A perturbed circular membrane immersed in a viscous incompressible fluid

$$\mathbf{X}(s) = (1 + \epsilon \cos ps)(\cos(s), \sin(s)), \quad p = 2$$

- Lagrangian force density $\mathbf{F} = K(t)\dfrac{\partial^2 \mathbf{X}}{\partial s^2}$, stiffness $K(t) = K_c(1 + \tau \sin(\omega_0 t))$

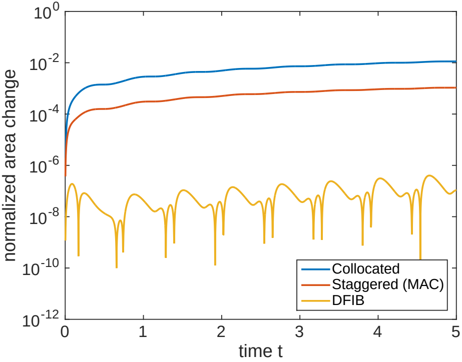
  $\Longrightarrow$ parametric resonance (Cortez et al. 2004)

- Standard IB method + collocated-grid fluid solver $\Longrightarrow$ substantial volume loss!



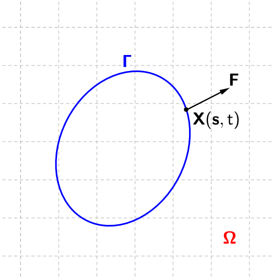
Work on improving volume conservation

- Modified discrete divergence operator (Peskin & Printz 1993)
- Staggered-grid fluid solver (Griffith 2012)
- Divergence-free velocity interpolation & force spreading (DFIB, Peskin)



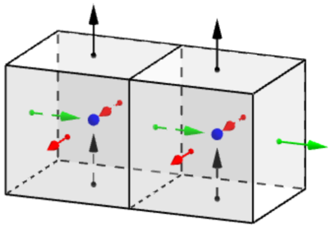
Equations of motion

$$
\begin{aligned}
\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) + \nabla p &= \mu\Delta\mathbf{v} + \mathbf{f} \\
\nabla\cdot\mathbf{v} &= 0 \\
\mathbf{f}(\mathbf{x},t) &= \int_{\boldsymbol{\Gamma}} \mathbf{F}(\mathbf{s},t)\delta(\mathbf{x}-\mathbf{X}(\mathbf{s},t))\,ds \\
\frac{\partial \mathbf{X}}{\partial t} &= \int_{\boldsymbol{\Omega}} \mathbf{v}(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{X}(\mathbf{s},t))\,dx \\
\mathbf{F}(\mathbf{s},t) &= \mathcal{F}(\mathbf{X}(\mathbf{s},t),t)
\end{aligned}
$$



- ▷ **Ω** : fluid domain, **x** : Eulerian
- ▷ **Γ** : structure, **s** : Lagrangian
- ▷ **F** : force density on structure
- ▷ Navier-Stokes + fluid-structure coupling via the δ-function

Spatial Discretization



- Periodic domain: $[0, L]^3$
  Meshwidth: $h = L/N$
- Staggered-grid discretization:
  Pressure p: cell-centered
  Velocity $\mathbf{u}$: face-centered

- Discrete partial derivative of grid function φ:

$$D^h_\alpha \varphi := \frac{\varphi(\mathbf{x} + \frac{h}{2}\mathbf{e}_\alpha) - \varphi(\mathbf{x} - \frac{h}{2}\mathbf{e}_\alpha)}{h}, \quad \alpha = 1, 2, 3.$$

  Discrete gradient: $\mathbf{D}^h \varphi := (D^h_1 \varphi,\ D^h_2 \varphi,\ D^h_3 \varphi)$

  Discrete divergence: $\mathbf{D}^h \cdot \mathbf{u} := \sum_{\alpha=1}^{3} D^h_\alpha u_\alpha,$

  Discrete Laplacian: $L^h \varphi = (\mathbf{D}^h \cdot \mathbf{D}^h)\varphi$

Spatial discretization

$$
\begin{aligned}
\rho\left(\frac{d\mathbf{u}}{dt}+\mathbf{N}(\mathbf{u})\right)+\mathbf{D}^h p &= \mu L^h\mathbf{u}+\mathbf{f}\\
\mathbf{D}^h\cdot\mathbf{u} &= 0\\
\mathbf{f} &= \mathcal{S}[\mathbf{X}]\mathbf{F}\\
\mathbf{U}_k &= \frac{d\mathbf{X}_k}{dt}=\mathcal{S}^*[\mathbf{X}_k]\mathbf{u}\\
\mathbf{F}_k &= \mathbf{F}(\mathbf{X}_k)=\mathcal{F}(\mathbf{X}_k,t)
\end{aligned}
$$

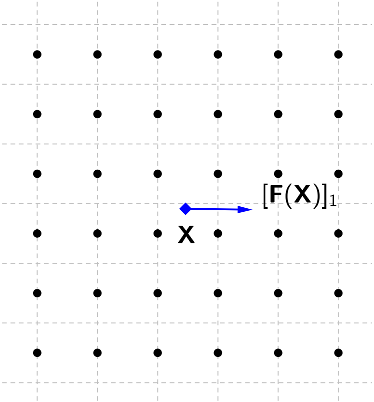
Lagrangian markers: $\mathbf{X}=\{\mathbf{X}_1,\ldots,\mathbf{X}_M\}$, $\mathbf{F}=\{\mathbf{F}_1,\ldots,\mathbf{F}_M\}$

- Standard IB force spreading: $\mathbf{f}(\mathbf{x})=\mathcal{S}[\mathbf{X}]\mathbf{F}=\sum_{k=1}^{M}\mathbf{F}_k\ \delta_h(\mathbf{x}-\mathbf{X}_k)\,\Delta\mathbf{s}_k$

  Standard IB velocity interpolation: $\mathbf{U}_k=\mathcal{S}^*[\mathbf{X}_k]\mathbf{u}=\sum_{\mathbf{x}}\mathbf{u}(\mathbf{x})\ \delta_h(\mathbf{x}-\mathbf{X}_k)h^3$

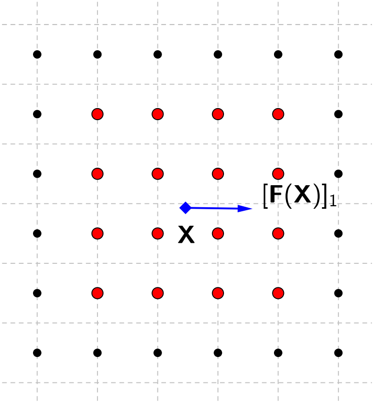
- Power identity (adjointness): $\sum_{\mathbf{x}}\mathbf{u}(\mathbf{x})\cdot\mathbf{f}(\mathbf{x})h^3=\sum_{k=1}^{M}\mathbf{F}_k\cdot\mathbf{U}_k$

Standard IB force spreading & velocity interpolation



- $\mathbf{u}$ divergence-free in the discrete sense: $\mathbf{D}^h \cdot \mathbf{u} = 0$
- $\mathbf{X}$: any arbitrary point in domain

Standard IB force spreading & velocity interpolation



- ▷ $\mathbf{u}$ divergence-free in the discrete sense: $\mathbf{D}^h \cdot \mathbf{u} = 0$
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  δ_h: 4-point discrete delta kernel

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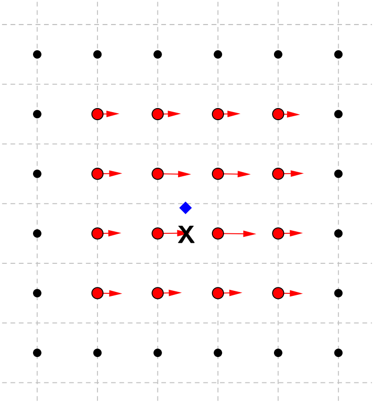

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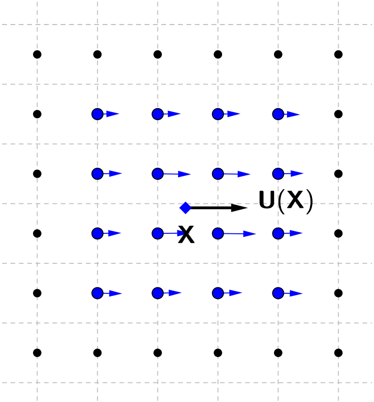
- Force spreading:

  $\mathbf{f}(\mathbf{x}) = \mathbf{F}\delta_h(\mathbf{x} - \mathbf{X})$

Standard IB force spreading & velocity interpolation



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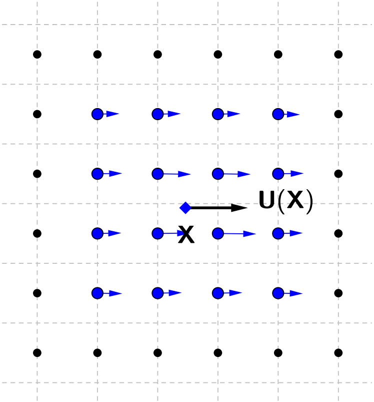
  $\mathbf{f}(\mathbf{x}) = \mathbf{F}\delta_h(\mathbf{x} - \mathbf{X})$

- Velocity interpolation:

  $\mathbf{U}(\mathbf{X}) = \sum_{\mathbf{x}} \mathbf{u}(\mathbf{x}) \; \delta_h(\mathbf{x} - \mathbf{X})h^3$

Standard IB force spreading & velocity interpolation



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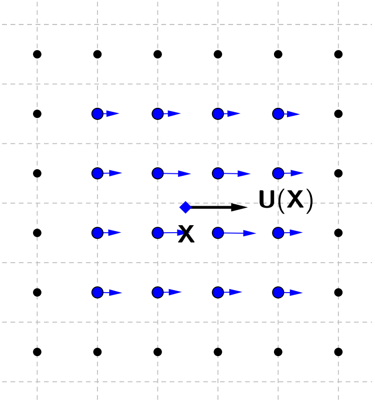
- spreading & interpolation: local!

Standard IB force spreading & velocity interpolation



- ▷ $\mathbf{u}$ divergence-free in the discrete sense: $\mathbf{D}^h \cdot \mathbf{u} = 0$
- ▷ $\mathbf{X}$: any arbitrary point in domain

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- ▷ Force spreading:

  $\mathbf{f}(\mathbf{x}) = \mathbf{F}\delta_h(\mathbf{x} - \mathbf{X})$

- ▷ Velocity interpolation:

  $\mathbf{U}(\mathbf{X}) = \sum_{\mathbf{x}} \mathbf{u}(\mathbf{x})\ \delta_h(\mathbf{x} - \mathbf{X})h^3$

- ▷ spreading & interpolation: local!

- ▷ $\mathbf{U}(\mathbf{X})$ is not divergence-free in the continuous sense,

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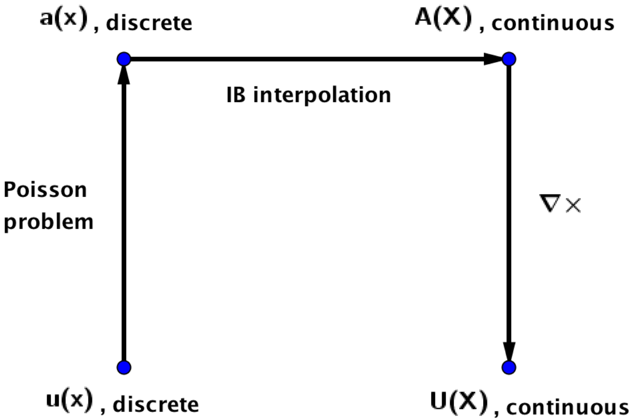
Divergence-free velocity interpolation

- **Challenge**: given $\mathbf{u}(\mathbf{x})$ that is discretely divergence-free, how to construct $\mathbf{U}(\mathbf{X})$ that is continuously divergence-free?
- **Idea**: $\mathbf{U}(\mathbf{X}) = (\nabla \times \mathbf{A})(\mathbf{X})$, where $\mathbf{A}(\mathbf{X})$ is the vector potential of $\mathbf{U}(\mathbf{X})$.



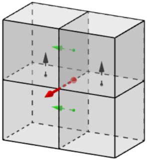
Divergence-free velocity interpolation

- First, construct a discrete vector potential $\mathbf{a}(\mathbf{x})$ by solving

$$\begin{cases} \mathbf{u} - \mathbf{u}_0 & = & \mathbf{D}^h \times \mathbf{a} \\ 0 & = & \mathbf{D}^h \cdot \mathbf{a} \end{cases} \implies -(\mathbf{D}^h \cdot \mathbf{D}^h)\mathbf{a} = \mathbf{D}^h \times \mathbf{u} \quad \text{(FFT!)} \qquad (1)$$

  where $\mathbf{u}_0$ is the mean of $\mathbf{u}$ and the gauge condition: $\mathbf{D}^h \cdot \mathbf{a} = 0$
- Question: where is $\mathbf{a}(\mathbf{x})$ defined on the staggered grid? edge-centered!

  For example, $[\mathbf{a}(\mathbf{x})]_1 = D_2^h u_3 - D_3^h u_2$



- Next, interpolate $\mathbf{a}(\mathbf{x})$ to get $\mathbf{A}(\mathbf{X})$ via standard IB interpolation,

$$\mathbf{A}(\mathbf{X}) = \sum_{\mathbf{x}} \mathbf{a}(\mathbf{x})\, \delta_h(\mathbf{x} - \mathbf{X}) h^3.$$

- Finally, $\mathbf{U}(\mathbf{X}) = \mathbf{u}_0 + (\nabla \times \mathbf{A})(\mathbf{X}) = \mathbf{u}_0 + \sum_{\mathbf{x}} \mathbf{a}(\mathbf{x}) \times (\nabla \delta_h)(\mathbf{x} - \mathbf{X}) h^3$.
- This ensures that $\nabla \cdot \mathbf{U} = 0$.

Divergence-free force spreading

- Goal: seek force spreading $\mathbf{f}$ that preserves the power identity

$$\sum_{\mathbf{x}} \mathbf{u}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) h^3 = \sum_{k=1}^{M} \mathbf{F}_k \cdot \mathbf{U}_k$$

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- $\sum_{\mathbf{x}} \mathbf{u}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})\, h^3 = \mathbf{u}_0 \cdot \sum_{\mathbf{x}} \mathbf{f}(\mathbf{x}) h^3 + \sum_{\mathbf{x}} (\mathbf{D}^h \times \mathbf{a})(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})\, h^3$

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- $$\sum_{\mathbf{x}} \mathbf{u}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})\, h^3 = \mathbf{u}_0 \cdot \sum_{\mathbf{x}} \mathbf{f}(\mathbf{x}) h^3 + \sum_{\mathbf{x}} (\mathbf{D}^h \times \mathbf{a})(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})\, h^3$$
$$= \mathbf{u}_0 \cdot \mathbf{f}_0 V + \sum_{\mathbf{x}} \mathbf{a}(\mathbf{x}) \cdot (\mathbf{D}^h \times \mathbf{f})(\mathbf{x})\, h^3,$$

  where $\mathbf{f}_0 = \frac{1}{V} \sum_{\mathbf{x}} \mathbf{f}(\mathbf{x})\, h^3$.

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- $$\sum_{k=1}^{M} \mathbf{U}_k \cdot \mathbf{F}_k = \mathbf{u}_0 \cdot \sum_{k=1}^{M} \mathbf{F}_k + \sum_{k=1}^{M} \sum_{\mathbf{x}} \mathbf{a}(\mathbf{x}) \times (\nabla \delta_h)(\mathbf{x} - \mathbf{X}_k) \cdot \mathbf{F}_k\, h^3$$

Divergence-free force spreading

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$$\sum_{\mathbf{x}} \mathbf{u}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) h^3 = \sum_{k=1}^{M} \mathbf{F}_k \cdot \mathbf{U}_k$$

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- $$\sum_{k=1}^{M} \mathbf{U}_k \cdot \mathbf{F}_k = \mathbf{u}_0 \cdot \sum_{k=1}^{M} \mathbf{F}_k + \sum_{k=1}^{M} \sum_{\mathbf{x}} \mathbf{a}(\mathbf{x}) \times (\nabla \delta_h)(\mathbf{x} - \mathbf{X}_k) \cdot \mathbf{F}_k\, h^3$$
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- want to set $\mathbf{f}_0 = \frac{1}{V} \sum_{k=1}^{M} \mathbf{F}_k$ and $(\mathbf{D}^h \times \mathbf{f})(\mathbf{x}) = \sum_{k=1}^{M} (\nabla \delta_h)(\mathbf{x} - \mathbf{X}_k) \times \mathbf{F}_k$

Divergence-free force spreading

- Goal: seek force spreading $\mathbf{f}$ that preserves the power identity

$$\sum_{\mathbf{x}} \mathbf{u}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) h^3 = \sum_{k=1}^{M} \mathbf{F}_k \cdot \mathbf{U}_k$$

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- The LHS term $\mathbf{D}^h \cdot (\mathbf{D}^h \times \mathbf{f}) = 0$, but the RHS term $\sum_{k=1}^{M} (\nabla \delta_h)(\mathbf{x} - \mathbf{X}_k) \times \mathbf{F}_k$ is not necessarily discretely divergence-free!

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- The LHS term $\mathbf{D}^h \cdot (\mathbf{D}^h \times \mathbf{f}) = 0$, but the RHS term $\sum_{k=1}^{M} (\nabla \delta_h)(\mathbf{x} - \mathbf{X}_k) \times \mathbf{F}_k$ is not necessarily discretely divergence-free!
- To fix this, we can add $\mathbf{D}^h \varphi$ on the RHS,

$$(\mathbf{D}^h \times \mathbf{f})(\mathbf{x}) = \sum_{k=1}^{M} (\nabla \delta_h)(\mathbf{x} - \mathbf{X}_k) \times \mathbf{F}_k + \mathbf{D}^h \varphi$$

becuase $\sum_{\mathbf{x}} \mathbf{a}(\mathbf{x}) \cdot \left(\mathbf{D}^h \varphi\right) h^3 = -\sum_{\mathbf{x}} \left(\cancelto{0}{\mathbf{D}^h \cdot \mathbf{a}}\right)(\mathbf{x})\, \varphi(\mathbf{x})\, h^3 = 0$

- It is also required to add $\mathbf{D}^h \varphi$ to guarantee the RHS to be discretely divergence-free.

Divergence-free force spreading

- $$(\mathbf{D}^h \times \mathbf{f})(\mathbf{x}) = \sum_{k=1}^{M} (\nabla \delta_h)(\mathbf{x} - \mathbf{X}_k) \times \mathbf{F}_k + \mathbf{D}^h \varphi$$
- Option 1: take $\mathbf{D}^h \cdot$ on both sides, solve a poisson equation of φ. unnecessary
- Option 2: take $\mathbf{D}^h \times$ on both sides,

$$\mathbf{D}^h \times (\mathbf{D}^h \times \mathbf{f}) = \mathbf{D}^h \times \left(\sum_{k=1}^{M} (\nabla \delta_h)(\mathbf{x} - \mathbf{X}) \times \mathbf{F}_k \right) + \cancelto{0}{\mathbf{D}^h \times (\mathbf{D}^h \varphi)}.$$

Gauge condition: $\mathbf{D}^h \cdot \mathbf{f} = 0$

$\Longrightarrow$

$$-(\mathbf{D}^h \cdot \mathbf{D}^h)\, \mathbf{f} = \mathbf{D}^h \times \left(\sum_{k=1}^{M} (\nabla \delta_h)(\mathbf{x} - \mathbf{X}_k) \times \mathbf{F}_k \right), \quad \text{(FFT!)}$$

- This only determines $\mathbf{f}$ up to a constant. For uniqueness, we use

$$\sum_{\mathbf{x}} \mathbf{f}(\mathbf{x})\, h^3 = \sum_{k=1}^{M} \mathbf{F}_k.$$

To summarize

- Given $\mathbf{u}$ that is discretely divergence-free, i.e., $\mathbf{D}^h \cdot \mathbf{u} = 0$

  Divergence-free velocity interpolation:

  1. $-(\mathbf{D}^h \cdot \mathbf{D}^h)\,\mathbf{a} = \mathbf{D}^h \times \mathbf{u}$
  2. $\mathbf{U}(\mathbf{X}) = \mathbf{u}_0 + \sum_{\mathbf{x}} \mathbf{a}(\mathbf{x}) \times (\nabla\delta_h)(\mathbf{x} - \mathbf{X})h^3$

  Divergence-free force spreading:

  1. $-(\mathbf{D}^h \cdot \mathbf{D}^h)\,\mathbf{f} = \mathbf{D}^h \times \left(\sum_{k=1}^{M} (\nabla\delta_h)(\mathbf{x} - \mathbf{X}_k) \times \mathbf{F}_k\right)$
  2. $\sum_{\mathbf{x}} \mathbf{f}(\mathbf{x})\, h^3 = \sum_{k=1}^{M} \mathbf{F}_k$

  Work added: two poisson problems (FFT!)

  New spreading & interpolation $\Longrightarrow$ global!

- Power identity: $\sum_{\mathbf{x}} \mathbf{u}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) h^3 = \sum_{k=1}^{M} \mathbf{F}_k \cdot \mathbf{U}_k$

A new C^3 6-point IB kernel (Peskin)

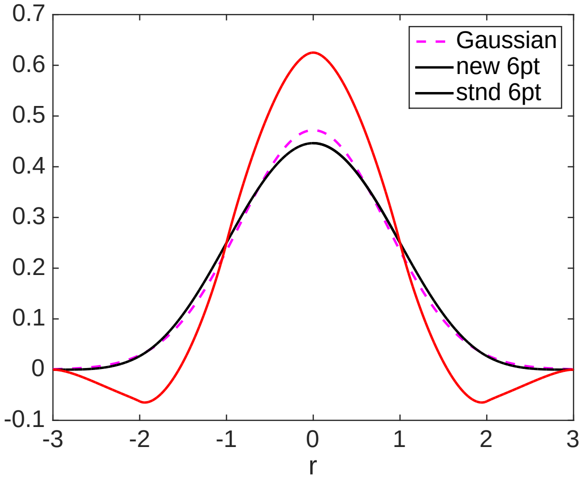
$$\delta_h(\mathbf{x}) = \frac{1}{h^3}\phi\left(\frac{x_1}{h}\right)\phi\left(\frac{x_2}{h}\right)\phi\left(\frac{x_3}{h}\right)$$

- Defining postulates for $\phi(r)$:

$$\begin{aligned}
&\text{(i)} && \phi(r) \text{ is continuous for all real } r,\\
&\text{(ii)} && \phi(r) = 0 \text{ for } |r| \geq 3,\\
&\text{(iii)} && \sum_{j \text{ even}} \phi(r-j) = \sum_{j \text{ odd}} \phi(r-j) = \frac{1}{2} \text{ for all real } r,\\
&\text{(iv)} && \sum_j (r-j)\,\phi(r-j) = 0,\\
&\text{(v)} && \sum_j (r-j)^2 \phi(r-j) = K,\\
&\text{(vi)} && \sum_j (r-j)^3 \phi(r-j) = 0,\\
&\text{(vii)} && \sum_j \left(\phi(r-j)\right)^2 = C.
\end{aligned}$$

- If $K = \frac{59}{60} - \frac{\sqrt{29}}{20} \Longrightarrow \phi$ has three continuous derivatives!

A new C^3 6-point IB kernel (Peskin)



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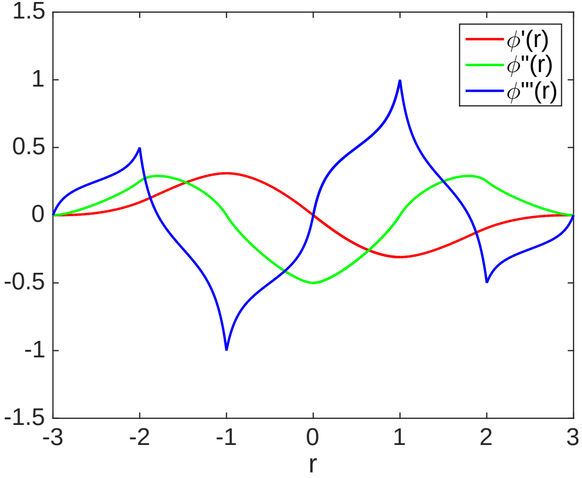
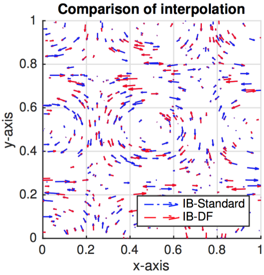


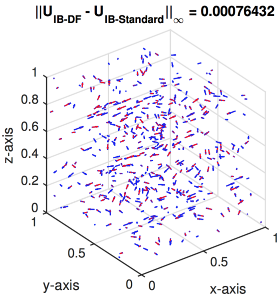
Test divergence-free velocity interpolation

- Taylor vortex:

$$\begin{aligned} v_1 &= 2\cos(2\pi x)\sin(2\pi y)\sin(2\pi z), \\ v_2 &= -\sin(2\pi x)\cos(2\pi y)\sin(2\pi z), \\ v_3 &= -\cos(2\pi x)\sin(2\pi y)\cos(2\pi z). \end{aligned}$$

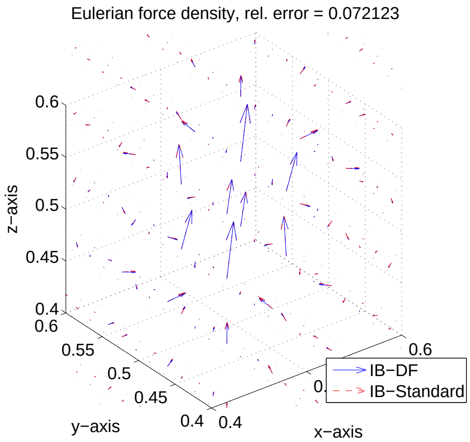
- $\mathbf{u} = \mathbb{P}\mathbf{v}$, where $\mathbb{P} = I - \mathbf{D}^h(\mathbf{D}^h \cdot \mathbf{D}^h)^{-1}\mathbf{D}^h$.
- interpolate $\mathbf{U}(\mathbf{X})$ at random $\mathbf{X}$ from $\mathbf{u}$





Test divergence-free force spreading

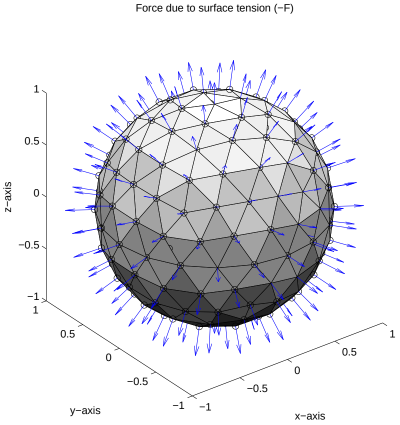
Spread a point force $\mathbf{F} = (1,1,1)$ at $X = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$



Test divergence-free force spreading

Spread surface tension on a triangulated sphere:

$$\mathbf{F}_k = -\frac{\partial \mathcal{E}}{\partial \mathbf{X}_k}, \text{ where } \mathcal{E}[\mathbf{X}_1, \cdots \mathbf{X}_M] = \sum_l A_l$$



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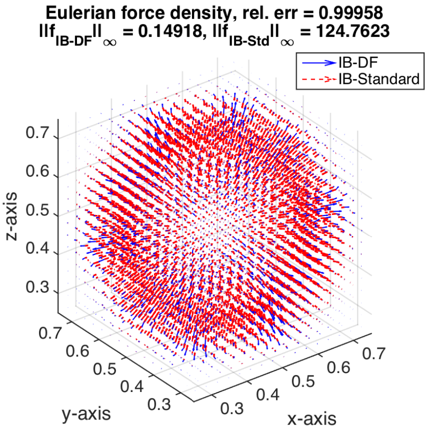
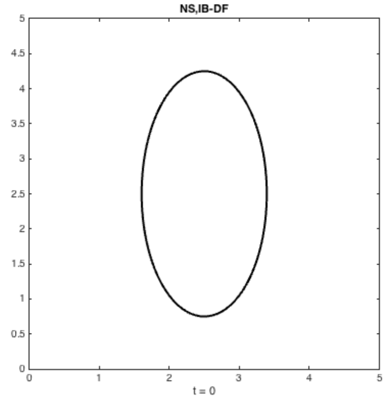


A 2D membrane with surface tension

- Setup: a 2D elliptical membrane immersed in a fluid at rest
- Surface tension: $\mathbf{F} = T\frac{\partial \tau}{\partial s}$, where $\tau = \frac{\partial \mathbf{X}/\partial s}{|\partial \mathbf{X}/\partial s|}$



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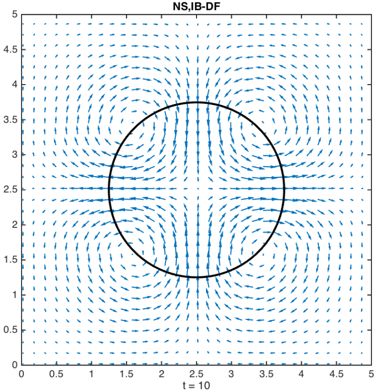
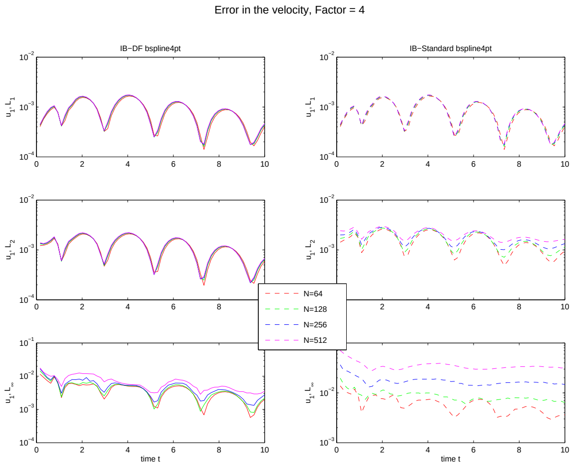
- Surface tension: $\mathbf{F} = T\frac{\partial \tau}{\partial s}$, where $\tau = \frac{\partial \mathbf{X}/\partial s}{|\partial \mathbf{X}/\partial s|}$
- Parameters: $\rho = 1, \mu = 0.1, N = 128, L = 5, h = L/N, dt = h/4, T = 2$



A 2D membrane with surface tension

- ▷ Setup: a 2D elliptical membrane immersed in a fluid at rest
- ▷ Surface tension: $\mathbf{F} = T\frac{\partial \tau}{\partial s}$, where $\tau = \frac{\partial \mathbf{X}/\partial s}{|\partial \mathbf{X}/\partial s|}$
- ▷ 2nd order convergence in $\mathbf{u}$



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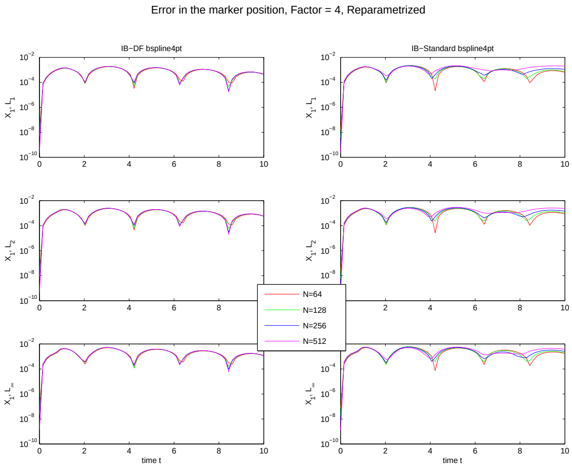
- 2nd order convergence in $\mathbf{X}$



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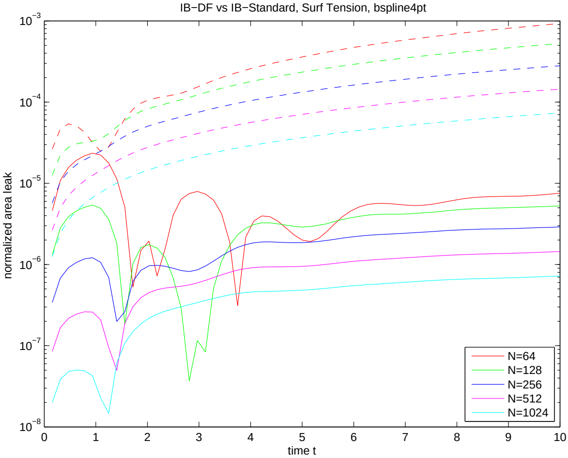
- Volume conservation



Marker spacing & volume conservation

- ▷ Setup: a 2D circular membrane (equilibrium) immersed in a fluid at rest
- ▷ Force density $\mathbf{F} = \frac{\partial^2 \mathbf{X}}{\partial s^2}$

Marker spacing & volume conservation

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- Analytic solution: $\mathbf{u} = 0$. Any spurious flow is numerical error!

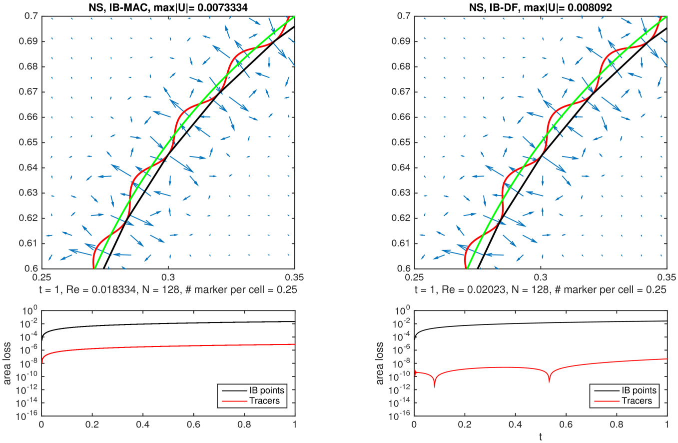
Marker spacing & volume conservation

- Setup: a 2D circular membrane (equilibrium) immersed in a fluid at rest
- Force density $\mathbf{F} = \frac{\partial^2 \mathbf{X}}{\partial s^2}$
- Analytic solution: $\mathbf{u} = 0$. Any spurious flow is numerical error!
- Investigate the relationship between marker spacing (# of makers per mesh-width) and volume conservation

Marker spacing & volume conservation

- Setup: a 2D circular membrane (equilibrium) immersed in a fluid at rest
- Force density $\mathbf{F} = \frac{\partial^2 \mathbf{X}}{\partial s^2}$

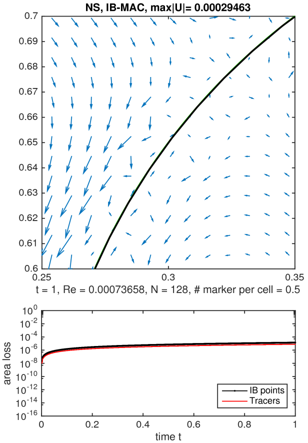
  0.25 marker per meshwidth (4 meshwidth between two IB markers)



Marker spacing & volume conservation

- Setup: a 2D circular membrane (equilibrium) immersed in a fluid at rest
- Force density $\mathbf{F} = \frac{\partial^2 \mathbf{X}}{\partial s^2}$

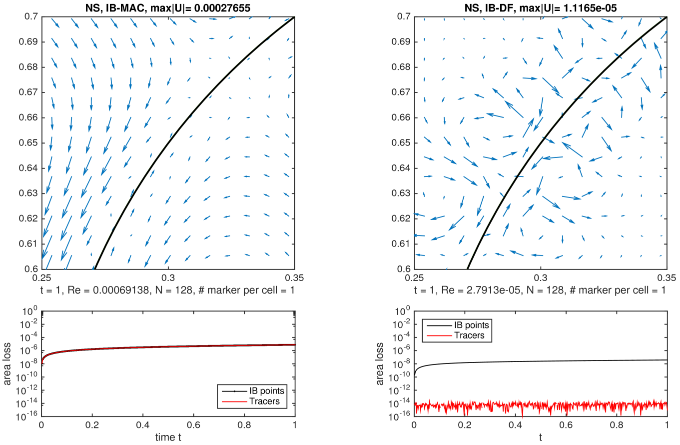
  0.5 marker per meshwidth (2 meshwidth between two IB markers)



Marker spacing & volume conservation

- Setup: a 2D circular membrane (equilibrium) immersed in a fluid at rest
- Force density $\mathbf{F} = \dfrac{\partial^2 \mathbf{X}}{\partial s^2}$

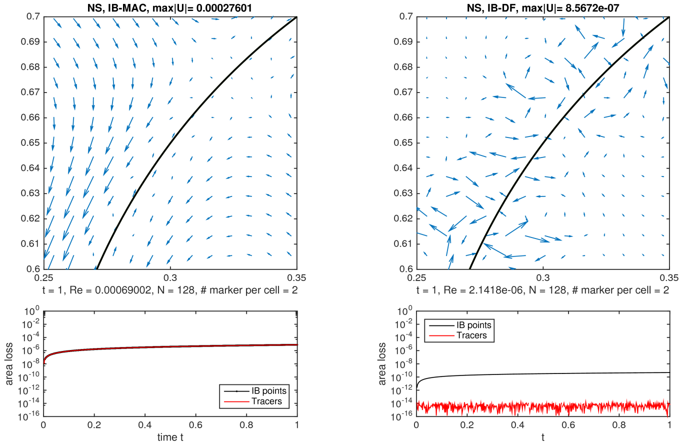
  1 marker per meshwidth



Marker spacing & volume conservation

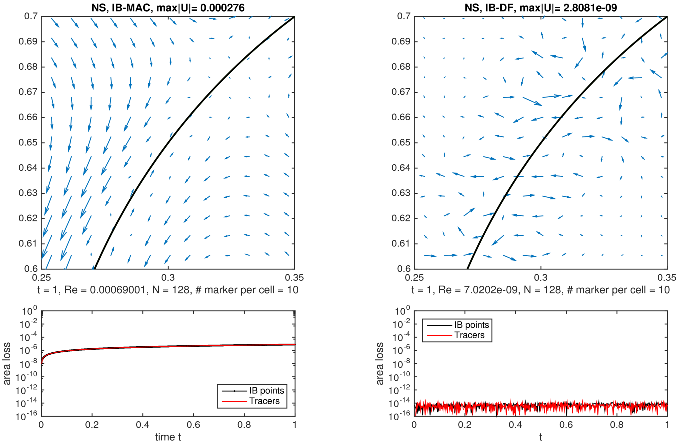
- Setup: a 2D circular membrane (equilibrium) immersed in a fluid at rest
- Force density $\mathbf{F} = \frac{\partial^2 \mathbf{X}}{\partial s^2}$

  2 markers per meshwidth



Marker spacing & volume conservation

- Setup: a 2D circular membrane (equilibrium) immersed in a fluid at rest
- Force density $\mathbf{F} = \dfrac{\partial^2 \mathbf{X}}{\partial s^2}$
  10 markers per meshwidth



In Closing

- Divergence-free velocity interpolation & force spreading

- Improved volume conservation & accuracy

- A new C^3 6-point discrete delta function

- Remark: discretely divergence-free force spreading $\mathbf{f}(\mathbf{x}) \Longrightarrow \mathbf{f}(\mathbf{x})$ includes the pressure gradient that is generated by Lagrangian forces. not yet clear how to isolate ∇p from force spreading (advantage or disadvantage?)