Floquet Theory for Internal Gravity Waves in a Density-Stratified Fluid

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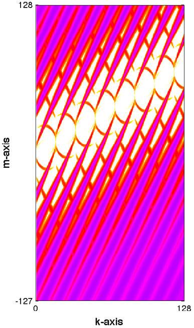
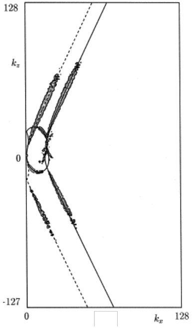
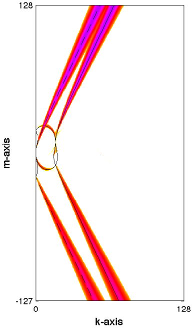






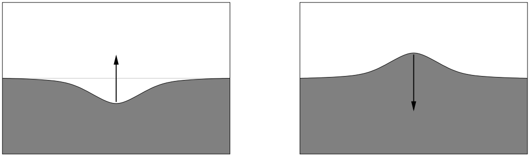
Density-Stratified Fluid Dynamics

Density-Stratified Fluids

- density of the fluid varies with altitude
  - stable stratification: heavy fluids below light fluids, internal waves
  - unstable stratification: heavy fluids above light fluids, convective dynamics

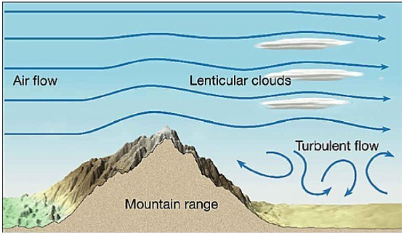
Buoyancy-Gravity Restoring Dynamics

- uniform stable stratification: $d\rho/dz < 0$ constant



- vertical displacements $\Rightarrow$ oscillatory motions

Internal Gravity Waves

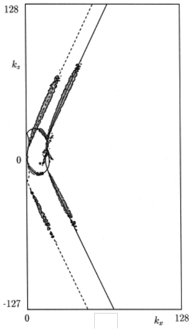


- evidence of internal gravity waves in the atmosphere
  - left: lenticular clouds near Mt. Ranier, Washington
  - right: uniform flow over a mountain $\Rightarrow$ oscillatory wave motions
- scientific significance of studying internal gravity waves
  - internal waves are known to be unstable
  - a major suspect of clear-air-turbulence

Gravity Wave Instability: Three Approaches

Triad resonant instability (Davis & Acrivos 1967, Hasselmann 1967)

- ▷ primary wave + 2 infinitesimal disturbances ⇒ exponential growth
- ▷ perturbation analysis



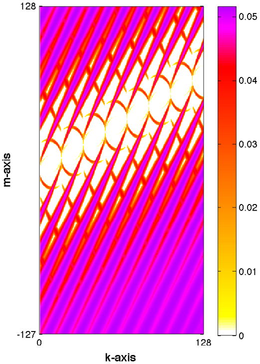
Direct Numerical Simulation (Lin 2000)

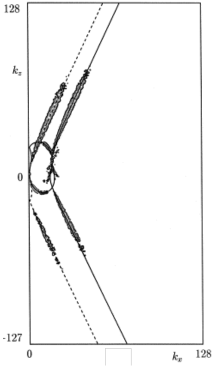
- ▷ primary wave + weak white-noise modes
- ▷ stability diagram
  - ▷ unstable Fourier modes

Linear Stability Analysis & Floquet-Fourier method (Mied 1976, Drazin 1977)

- ▷ linearized Boussinesq equations & stability via eigenvalue computation

My Thesis Goal





- ▷ Floquet-Fourier computation: over-counting of instability in wavenumber space
- ▷ Lin's DNS: two branches of disturbance Fourier modes
- ▷ goal: <u>to identify all physically unstable modes from Floquet-Fourier computation</u>

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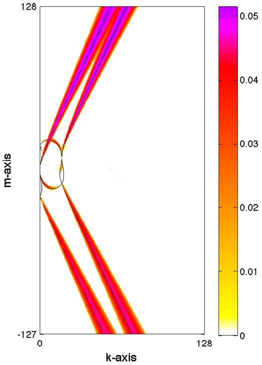

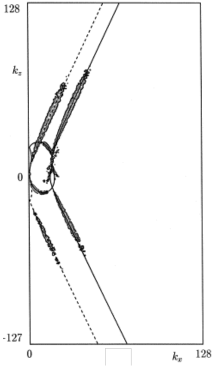

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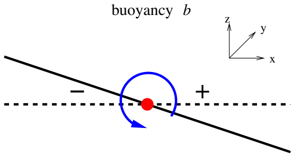
The Governing Equations

Boussinesq Equations in Vorticity-Buoyancy Form

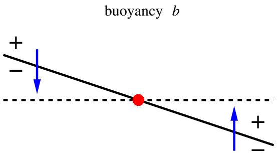
$$\boxed{\nabla \cdot \vec{u} \;=\; 0 \quad ; \quad \frac{D\eta}{Dt} \;=\; -b_x \quad ; \quad \frac{Db}{Dt} \;=\; -\mathcal{N}^2 w}$$

- incompressible, inviscid Boussinesq Fluid
  - Euler equations + weak density variation (the Boussinesq approximation)
  - Brunt-Vaisala frequency $\mathcal{N}$: uniform stable stratification, $\mathcal{N}^2 > 0$
- 2D velocity: $\vec{u}(x,z,t)$; buoyancy: $b(x,z,t)$
  - streamfunction: $\vec{u} = \begin{pmatrix} u \\ w \end{pmatrix} = -\vec{\nabla} \times \psi\,\hat{y} = \begin{pmatrix} -\psi_z \\ \psi_x \end{pmatrix}$
  - vorticity: $\vec{\nabla} \times \vec{u} = \eta\,\hat{y} = \nabla^2 \psi\,\hat{y}$

Exact Plane Gravity Wave Solutions



$$\frac{D\eta}{Dt} = -b_x$$



$$\frac{Db}{Dt} = -\mathcal{N}^2 w$$

- ▷ dynamics of buoyancy & vorticity $\Rightarrow$ oscillatory wave motions
- ▷ exact plane gravity wave solutions

$$\begin{pmatrix} \psi \\ b \\ \eta \end{pmatrix} = \begin{pmatrix} -\Omega_d/K \\ \mathcal{N}^2 \\ \mathcal{N}^2 K/\Omega_d \end{pmatrix} 2\mathcal{A}\,\sin(Kx + Mz - \Omega_d t)$$

  - ▷ primary wavenumbers: (K, M)
  - ▷ dispersion relation: $\Omega_d^2(K, M) = \dfrac{\mathcal{N}^2 K^2}{K^2 + M^2}$.

Linear Stability Analysis

- dimensionless exact plane wave + small disturbances

$$\begin{pmatrix} \psi \\ b \\ \eta \end{pmatrix} = \begin{pmatrix} -\Omega \\ 1 \\ 1/\Omega \end{pmatrix} 2\epsilon \sin(x+z-\Omega t) + \begin{pmatrix} \tilde{\psi} \\ \tilde{b} \\ \tilde{\eta} \end{pmatrix}$$

  - ϵ: dimensionless amplitude & dimensionless frequency: $\Omega^2 = \dfrac{1}{1+\delta^2}$

- linearized Boussinesq equations

$$\boxed{\begin{array}{ccccccc} & & & \delta^2\tilde{\psi}_{xx} + \tilde{\psi}_{zz} & = & \tilde{\eta} \\ \tilde{\eta}_t & + & \tilde{b}_x & - & 2\epsilon J(\Omega\tilde{\eta} + \tilde{\psi}/\Omega\,,\, \sin(x+z-\Omega t)) & = & 0 \\ \tilde{b}_t & - & \tilde{\psi}_x & - & 2\epsilon J(\Omega\tilde{b} + \tilde{\psi}\,,\, \sin(x+z-\Omega t)) & = & 0 \end{array}}$$

  - $\delta = K/M$: related to the wave propagation angle (Lin: $\delta = 1.7$)
  - Jacobian determinant

$$J(f,g) = \begin{vmatrix} f_x & g_x \\ f_z & g_z \end{vmatrix} = f_x g_z - g_x f_z$$

Linear Stability Analysis

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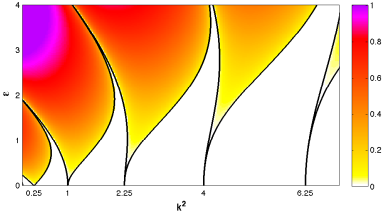
  - system of linear PDEs with non-constant, but periodic coefficients
  - analyzed by Floquet theory
  - classical textbook example: Mathieu equation (Chapter 3)

Floquet Theory: Mathieu Equation

Mathieu Equation:

$$\boxed{\frac{d^2u}{dt^2} + \left[k^2 - 2\epsilon\cos(t)\right] u = 0}$$



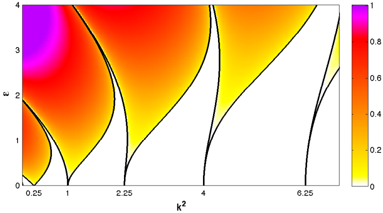
- second-order linear ODE with periodic coefficients
- Floquet theory: $u = e^{-i\omega t} \cdot p(t) =$ exponential part $\times$ co-periodic part
- Floquet exponent $\omega(k;\epsilon)$: $\mathrm{Im}\,\omega > 0 \rightarrow$ instability

- goal: to identify all unstable solutions in (k, ϵ)-space

Floquet Theory: Mathieu Equation

Mathieu Equation:

$$\frac{d^2u}{dt^2} + \left[k^2 - 2\epsilon\cos(t)\right]u = 0$$



Two perspectives:

- perturbation analysis $\Rightarrow$ two branches of Floquet exponent
  - away from resonances: $\omega(k;\epsilon) \sim \pm k$
  - resonant instability at primary resonance $\left(k = \frac{1}{2}\right)$: $\omega(k;\epsilon) \sim \pm\frac{1}{2} + i\,\epsilon$
- Floquet-Fourier computation of $\omega(k;\epsilon)$
  - a Riemann surface interpretation of $\omega(k;\epsilon)$ with $k \in \mathbb{C}$

Floquet-Fourier Computation

- Mathieu equation in system form:

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = i\begin{bmatrix} 0 & 1 \\ k^2 - 2\epsilon\cos(t) & 0 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

- Floquet-Fourier representation:

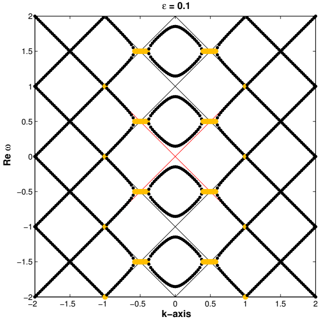
$$\begin{pmatrix} u \\ v \end{pmatrix} = e^{-i\omega t}\cdot\sum_{m=-\infty}^{\infty}\vec{c}_m e^{-imt}$$

- $\omega(k;\epsilon)$ as eigenvalues of Hill's bi-infinite matrix:

$$\begin{bmatrix} \ddots & \ddots & & \\ \ddots & \mathbf{S}_0 & \epsilon\mathbf{M} & \\ & \epsilon\mathbf{M} & \mathbf{S}_1 & \ddots \\ & & \ddots & \ddots \end{bmatrix}$$

  - 2×2 real blocks: $\mathbf{S}_m$ and $\mathbf{M}$
- truncated Hill's matrix: $-N \le m \le N$
  - real-coefficient characteristic polynomial
  - compute $4N+2$ eigenvalues: $\{\omega_n(k;\epsilon)\}$
    - $\epsilon = 0$, eigenvalues from $\mathbf{S}_n$ blocks: $\omega_n(k;0) = -n \pm k$ & all real-valued
    - $\epsilon \ll 1$, complex eigenvalues may arise from $\epsilon = 0$ double eigenvalues

Floquet-Fourier Computation

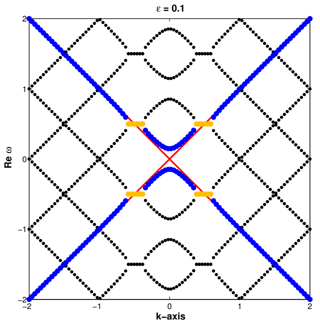


- $\omega_n(k;\epsilon)$: real ● ; complex ●
- '——': $\omega_n(k;0) = -n \pm k$
  '——': $\omega_0(k;0) = \pm k$
- $\omega_n(k;\epsilon)$ curves are close to $\omega_n(k;0)$
- two continuous curves close to $\pm k$
- the rest are shifted due to

$$\begin{pmatrix} u \\ v \end{pmatrix} = e^{-i(\omega_0+n)t} \cdot \sum_{m=-\infty}^{\infty} \vec{c}_{m+n} e^{-imt}$$

- For each k, how many Floquet exponents are associated with the unstable solutions of Mathieu equation? two or ∞? Both!
  - two is understood from perturbation analysis
  - ∞ will be understood from the Riemann surface of $\omega(k;\epsilon)$ with $k \in \mathbb{C}$

Floquet-Fourier Computation



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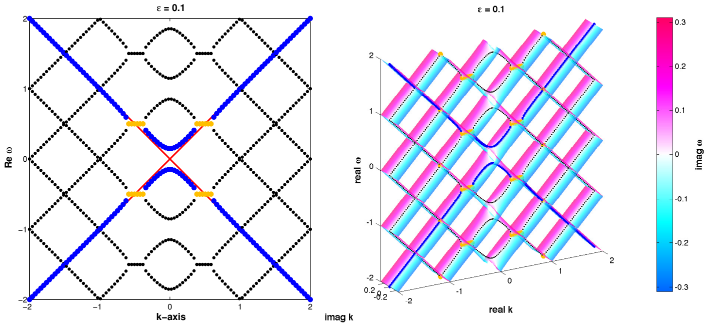
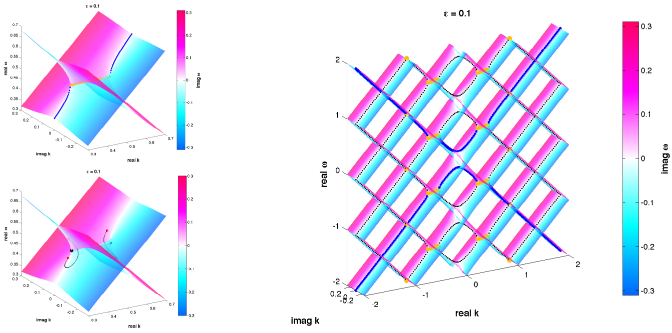
A Riemann Surface Interpretation of $\omega(k;\epsilon)$



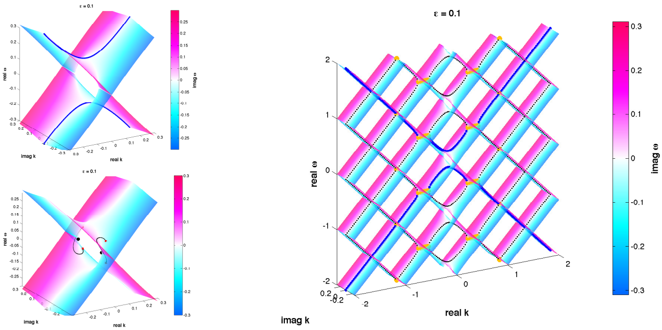
- Floquet-Fourier computation with $k \in \mathbb{C} \rightarrow$ the Riemann surface of $\omega(k;\epsilon)$
  - surface height: real ω ; surface colour: imag ω
  - layers of curves for $k \in \mathbb{R}$ become layers of sheets for $k \in \mathbb{C}$
  - the two physical branches belong to two primary Riemann sheets
- How to identify the two primary Riemann sheets?
  - more understanding of how sheets are connected

A Riemann Surface Interpretation of $\omega(k; \epsilon)$



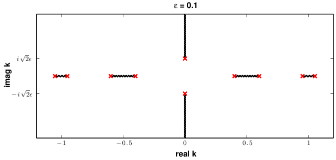
- ▷ zoomed view near $\text{Re}\, k = 1/2$ shows Riemann sheet connection
- ▷ branch points: end points of instability intervals
  - ▹ loop around the branch points $\Rightarrow \sqrt{\ }$ type
- ▷ branch cuts coincide with instability intervals (McKean & Trubowitz 1975)

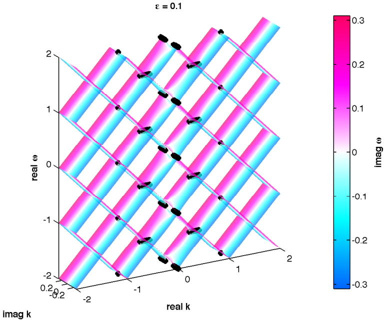
A Riemann Surface Interpretation of $\omega(k;\epsilon)$



- zoomed view near $\operatorname{Re} k = 0$ shows Riemann sheet connection
- branch points: two on imaginary axis
  - loop around the branch points $\Rightarrow \sqrt{\ }$ type
- branch cuts to $\pm i\infty$ give V-shaped sheets

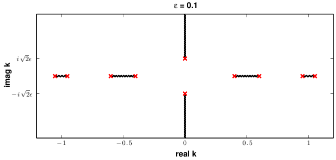
A Riemann Surface Interpretation of $\omega(k;\epsilon)$

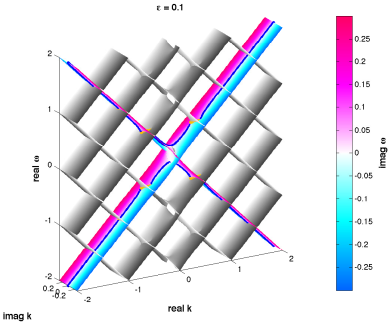




- ▷ branch cuts: instability intervals & two cuts to $\pm i\infty$
- ▷ two primary sheets: upward & downward V-shaped sheets
  - ▷ associated with the two physically-relevant Floquet exponents
  - ▷ the other sheets are integer-shifts of primary sheets

A Riemann Surface Interpretation of $\omega(k;\epsilon)$





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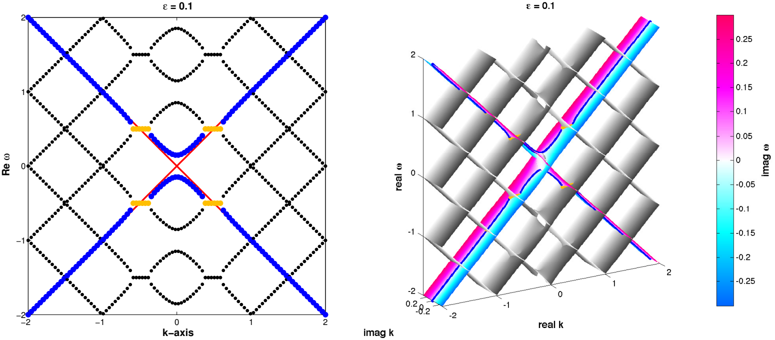
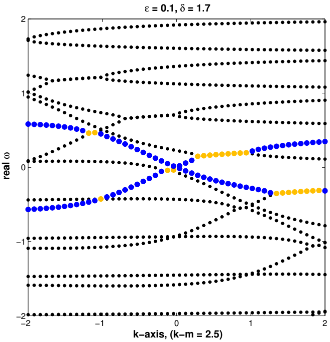
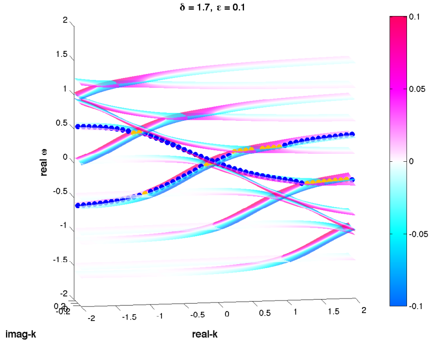
Recap of Mathieu Equation

- Floquet-Fourier: $\begin{pmatrix} u \\ v \end{pmatrix} = e^{-i\omega t} \cdot \sum_{m=-N}^{N} \vec{c}_n e^{-imt}$
  - $4N + 2$ computed Floquet exponents $\omega_n(k; \epsilon)$
- perturbation analysis: $\omega(k; \epsilon) \sim \pm k$
- Riemann surface has two primary Riemann sheets (physically-relevant)



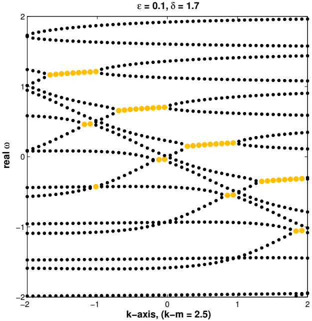
- Floquet-Fourier: $\begin{pmatrix} \tilde{\psi} \\ \tilde{b} \end{pmatrix} = e^{i(kx+mz-\omega t)} \cdot \left\{ \sum_{n=-N}^{N} \begin{pmatrix} \hat{\psi}_n \\ \hat{b}_n \end{pmatrix} e^{in(x+z-\Omega t)} \right\}.$
  - $4N+2$ computed Floquet exponents $\omega_n(k, m; \epsilon, \delta)$
- perturbation analysis: $\omega(k, m; \epsilon, \delta) \sim \pm \frac{|k|}{\sqrt{\delta^2 k^2 + m^2}}$
- Riemann surface analysis $\Rightarrow$ physically-relevant Floquet exponents

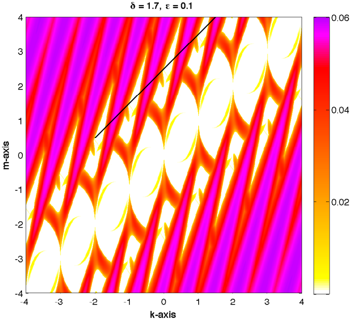




Gravity Wave Stability Problem

- four parameters of $\omega(k, m; \epsilon, \delta)$
  - ϵ, $\delta = 1.7$ (Lin)
  - wavevector, (k, m) ; $\quad k \in \mathbb{C}$ with $k - m = 2.5$
- over-counting of Floquet-Fourier computation
  - vertical & horizontal shifts $\quad \rightarrow \quad$ instability bands





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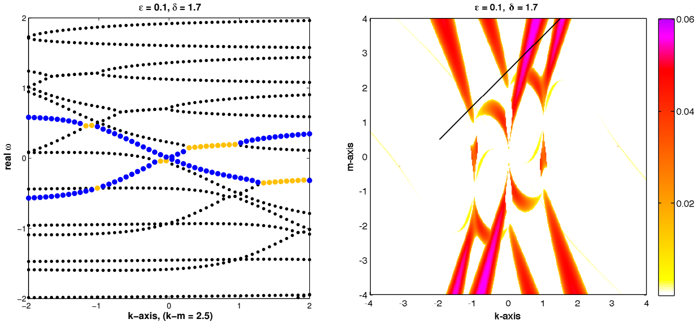
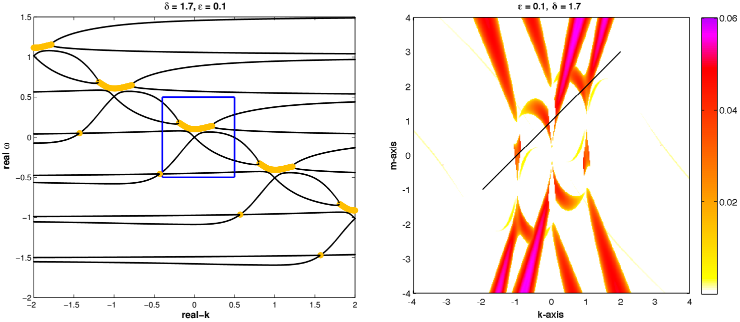


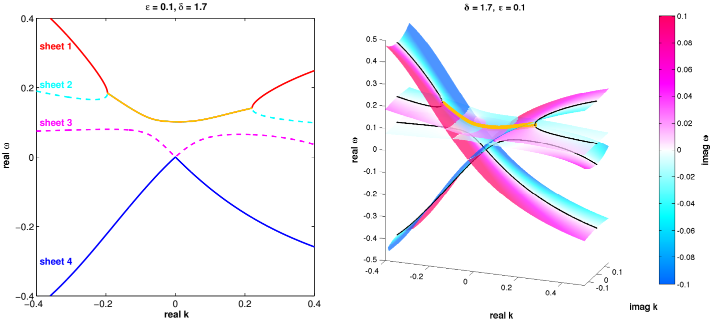
- physically-relevant Floquet exponents solves over-counting problem

Fixing the Gap along $k - m = 1$



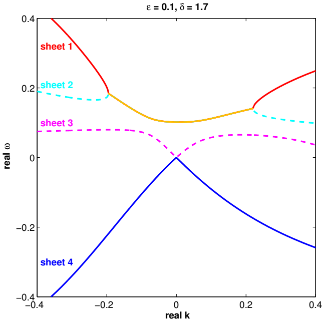
- ▷ new feature: four-sheet collision (only two for Mathieu!)
- ▷ physically corresponds to near-resonance of four fourier modes (section 5.3)

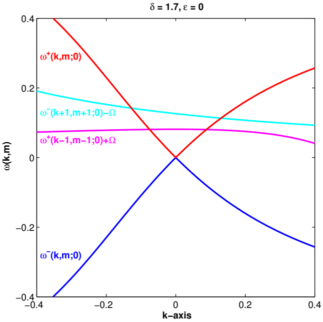
Fixing the Gap along $k - m = 1$



- ▷ zoomed view near $\mathrm{Re}\, k = 0$ with Riemann surface

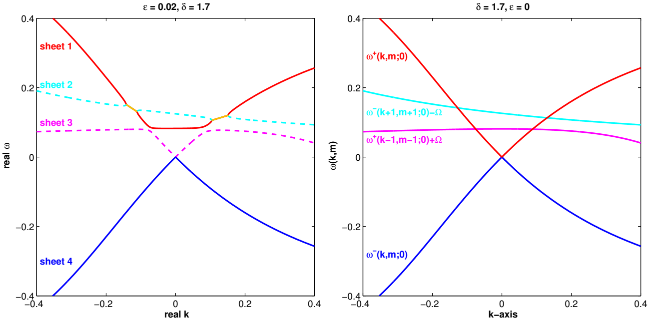
Fixing the Gap along $k - m = 1$





- ▷ continuation algorithm for $\omega(k, m; \epsilon = 0.1)$ starts from $\epsilon = 0$ values

Fixing the Gap along $k - m = 1$



- continuation algorithm for $\omega(k, m; \epsilon = 0.1)$ starts from $\epsilon = 0$ values
  - $\epsilon = 0.02$: shows $\epsilon = 0$ limit incorrect

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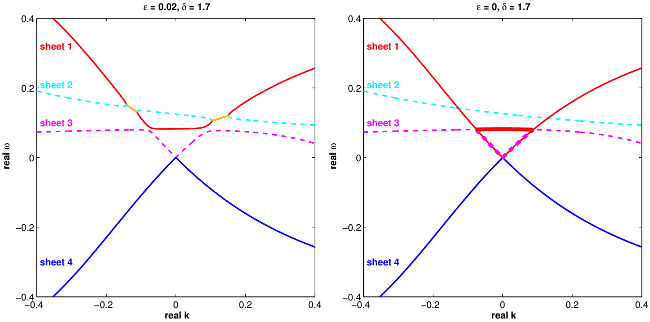
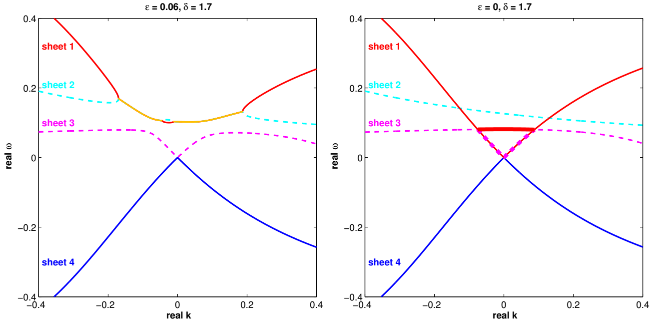


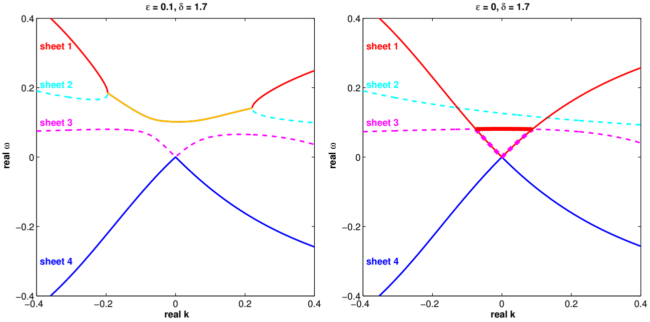
- continuation algorithm for $\omega(k, m; \epsilon = 0.1)$ starts from $\epsilon = 0$ values
  - $\epsilon = 0.02$: suggests redefining $\epsilon = 0$ branch values (continuous)

Fixing the Gap along $k - m = 1$



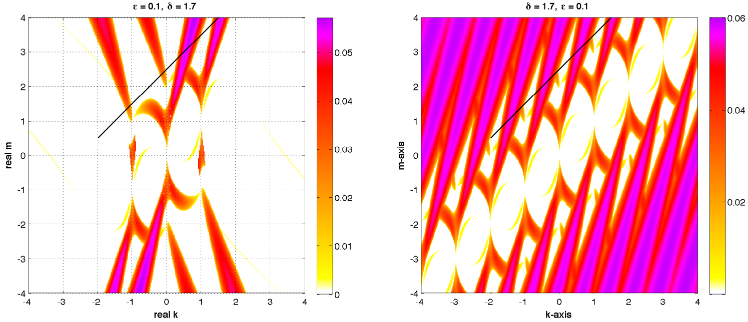
- continuation algorithm for $\omega(k, m; \epsilon = 0.1)$ starts from $\epsilon = 0$ values
  - $\epsilon = 0.06$: instability bands are about to merge

Fixing the Gap along $k - m = 1$



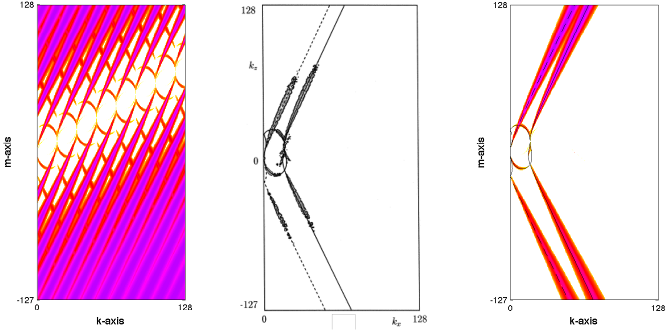
- continuation algorithm for $\omega(k, m; \epsilon = 0.1)$ starts from $\epsilon = 0$ values
  - $\epsilon = 0.1$: the gap is fixed

Instabilities from Two Primary Sheets



- ▷ stability diagram is a superposition of instabilities from the two primary sheets
- ▷ both primary sheets are continuous in $\mathrm{Re}\,\omega$ & $\mathrm{Im}\,\omega$
- ▷ <u>over-counting problem is solved by complex analysis!</u>

In Closing: What I Have Learned



- ▷ density-stratified fluid dynamics & internal gravity waves
- ▷ linear stability analysis
- ▷ the Mathieu equation, Floquet theory & Floquet-Fourier computation
- ▷ perturbation analysis (near & away from resonance)
- ▷ understanding the Riemann surface structure & computation

Four Sheets: $\epsilon = 0.1$